Quasi-Random Ideas. By Josef Dick.

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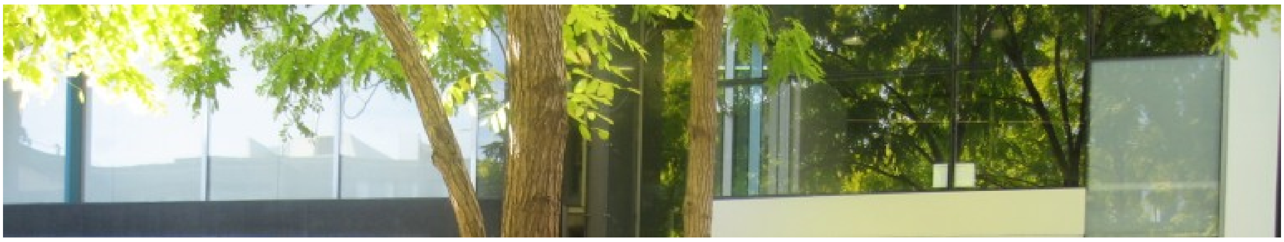

← Math2111: Chapter 3: Additional Material: Rectifiable parameterised curves

Math2111: Chapter 5: Additional Material: Differential forms and the general Stokes' theorem

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In this blog entry you can find lecture notes for Math2111, several variable calculus. See also the table of contents for this course. This blog entry printed to pdf is available here.

We have studied a variety of generalisations of the fundamental theorem of calculus:

- the fundamental theorem of line integrals;
- Green's theorem;
- Stokes' theorem;
- Gauss' Divergence theorem;

Now we show how all of these formulae concisely fit into one approach. Differential forms provide the underlying theory to present all formulae in one framework. This also allows us to generalise the theorems above to arbitrary dimensions.

Differential 1-forms

Consider an open set $\Omega \subset \mathbb{R}^3$ (more generally one could also consider open subsets of $\mathbb{R}^n$). To start, a differential 0 form is just a function $f:\Omega \to \mathbb{R}$.

A differential 1-form is an expression of the form

$$F_1(x,y,z)\,\mathrm{d}x + F_2(x,y,z)\,\mathrm{d}y + F_3(x,y,z)\,\mathrm{d}z,$$

where $F_1, F_2, F_3 : \Omega \to \mathbb{R}$. We have seen such expressions previously when we studied line integrals. If $\omega = F_1\,\mathrm{d}x + F_2\,\mathrm{d}y + F_3\,\mathrm{d}z$, and the curve $\mathcal{C}$ is parameterised by $\boldsymbol{c}(t) = (x(t), y(t), z(t))$, then the line integral of the vector field $\boldsymbol{F} = F_1\widehat{\boldsymbol{i}} + F_2\widehat{\boldsymbol{j}} + F_3\widehat{\boldsymbol{k}}$ over the curve $\mathcal{C}$ can be written as

$$\int_{\mathcal{C}} F_1\,\mathrm{d}x + F_2\,\mathrm{d}y + F_3\,\mathrm{d}z = \int_{\mathcal{C}} \omega.$$

As an example, let $f : \mathbb{R}^3 \to \mathbb{R}$ be continuously differentiable. Then

$$\mathrm{d}f = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y + \frac{\partial f}{\partial z}\,\mathrm{d}z$$

is a differential 1-form. It is called the *total differential* of f. If x, y, z are functions of t, then

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial x}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial f}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}.$$

Hence by cancelling $\mathrm{d}t$ in this formula one obtains the total differential of f.

Using this notation, we can now write the fundamental theorem of line integrals in a very concise form. Let the curve C be parameterised by $\boldsymbol{c}(t) = (x(t), y(t), z(t))$, where $a \le t \le b$. Then fundamental theorem of line integral states that

$$\int_C \nabla f \cdot \mathrm{d}s = \int_C \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y + \frac{\partial f}{\partial z}\,\mathrm{d}z = f(\boldsymbol{c}(b)) - f(\boldsymbol{c}(a)).$$

Using differential forms, this can be written as

$$\int_C \mathrm{d}f = f(\boldsymbol{c}(b)) - f(\boldsymbol{c}(a)).$$

Notice the similarity between this notation and the one dimensional integral $\int_{[a,b]} \,\mathrm{d}t = b - a$ (here we write $\int_{[a,b]}$ which just stands for $\int_a^b$).

Definition
Let $\omega = F_1\,\mathrm{d}x + F_2\,\mathrm{d}y + F_3\,\mathrm{d}z$ be a differential 1-form. If there is a function f such that $\omega = \mathrm{d}f$, then the differential form ω is called *exact*.

Let $\omega = F_1\,\mathrm{d}x + F_2\,\mathrm{d}y + F_3\,\mathrm{d}z$ be a differential 1-form. Then we can associate with it a vector field $\boldsymbol{F} = F_1\widehat{\boldsymbol{\imath}} + F_2\widehat{\boldsymbol{\jmath}} + F_3\widehat{\boldsymbol{k}}$. Then, $\boldsymbol{F}$ is conservative if there is a function f such that $\boldsymbol{F} = \nabla f$.

Hence, a differential 1-form is exact if and only if the associated vector field $\boldsymbol{F}$ is conservative. If the region where $\boldsymbol{F}$ is defined is simply connected, then we have seen that $\boldsymbol{F}$ is conservative if and only if $\mathrm{curl}\ \boldsymbol{F} = \boldsymbol{0}$. Hence we also get a criterion to check whether a differential 1-form is exact and a method of calculating a function f such that $\omega = \mathrm{d}f$. The details are left as an exercise.

Exact differential forms also appear in ordinary differential equations (ode). Consider the ode

$$F_1(x, y) + F_2(x, y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

By multiplying with $\mathrm{d}x$ and defining the differential $\omega = F_1\,\mathrm{d}x + F_2\,\mathrm{d}y$, the ode can be written as

$$\omega = 0.$$

If ω is an exact differential, then there is a function $f : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$ such that $\omega = \mathrm{d}f$. Hence, an exact ode can be written as

$$\mathrm{d}f = 0$$

where f is a function defined on a subset of $\mathbb{R}^2$. By integrating on both sides we get a solution $f(x, y) = k$, where k is a constant.

Differential 1-forms are useful for describing line integrals and gradients. But for double integrals, surface integrals, Green's theorem and Stokes' theorem we need a more general concept.

Bivectors and differential 2-forms

We now consider differential 2-forms, which can partly be understood geometrically.

Recall that, under certain conditions, Green's theorem in tangential form states that

$$\int_C F_1 \, \mathrm{d}x + F_2 \mathrm{d}y = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \, \mathrm{d}x \, \mathrm{d}y.$$

Leaving out the integral signs, then the left-hand side is again just a differential 1-form. The aim now is to investigate, given a differential 1-form, how can we obtain the formula on the right-hand side in Green's theorem? Once we know the underlying principles, then we can obtain analogous results in other situations.

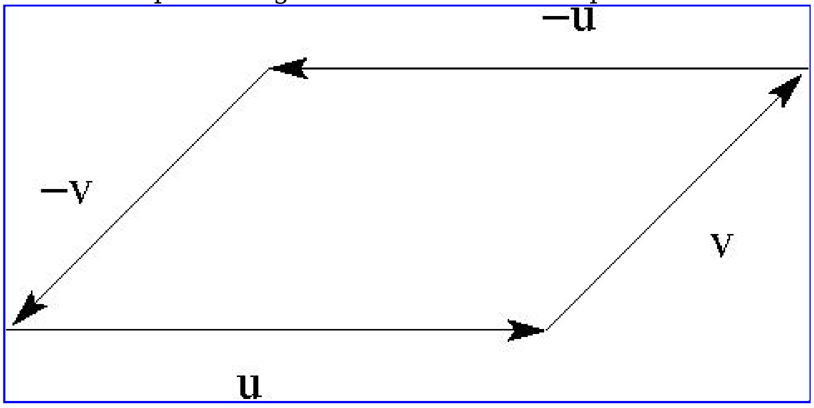
We have seen this principle already above. If f is a differential 0-form, then we can `apply $d$' to obtain a differential $1$-form $\mathrm{d}f$. Similarly, given a differential $1$-form $\omega = F_1 \, \mathrm{d}x + F_2 \, \mathrm{d}y$, we would again want to `apply d' (what this means will be made more precise below) to obtain a differential 2-form $\mathrm{d}\omega$, such that we can express Green's theorem as

$$\int_{\partial \Omega} \omega = \iint_\Omega \mathrm{d}\omega,$$

where $\Omega \subset \mathbb{R}^2$ is a suitable region and $\partial\Omega$ is its boundary.

Before we can describe how the operator d works on differential forms, let us study differential 2-forms in more detail.

Above we associated a differential 1-form with a vector. Differential 2-forms can be associated with bivectors (see also multivectors). For $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^3$ we define the bivector $\boldsymbol{u} \wedge \boldsymbol{v}$ (the symbol $\wedge$ is pronounced `wedge' and we call $\boldsymbol{u} \wedge \boldsymbol{v}$ the wedge product of $\boldsymbol{u}$ and $\boldsymbol{v}$). Such a bivector can be associated with the cross product $\boldsymbol{u} \times \boldsymbol{v}$. Geometrically, the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ span a parallelogram in space. This parallelogram lies in a plane (defined by the vectors $\boldsymbol{u}, \boldsymbol{v}$) and the length of the cross product $\boldsymbol{u} \times \boldsymbol{v}$ is the area of the parallelogram spanned by $\boldsymbol{u}$ and $\boldsymbol{v}$. Additionally, the cross product $\boldsymbol{u} \times \boldsymbol{v}$ also has a direction, which is determined by the right-hand rule. For bivectors, we can associate this direction with a direction (or orientation) of the boundary curve of the parallelogram. Hence we can represent a bivector $\boldsymbol{u} \wedge \boldsymbol{v}$ by the following picture:



Bivectors can be manipulated in the following ways. For a scalar $a \in \mathbb{R}$ we define the bivector $a(\boldsymbol{u} \wedge \boldsymbol{v})$, whose associated parallelogram has area a times $\mathrm{Area}(\boldsymbol{u} \wedge \boldsymbol{v})$. If $a>0$, then the parallelogram has the same orientation and if $a<0$, then the orientation is reversed. The following statements can be interpreted using the parallelogram analogy:

- changing the order of the vectors changes the orientation of the parallelogram, hence:

$$\boldsymbol{u} \wedge \boldsymbol{v} = -\boldsymbol{v} \wedge \boldsymbol{u}$$

- the area of a parallelogram spanned by $\boldsymbol{u}$ and $\boldsymbol{u}$ is zero, hence:

$$\boldsymbol{u} \wedge \boldsymbol{u} = 0$$

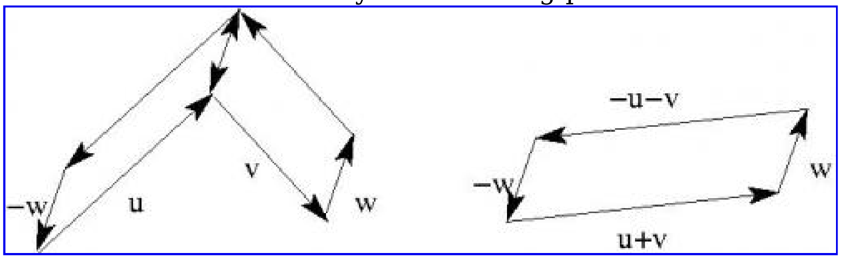
- a times the area of the parallelogram spanned by $\boldsymbol{u}$ and $\boldsymbol{v}$ is equal to the area of the parallelogram spanned by $a\boldsymbol{u}$ and $\boldsymbol{v}$ and is equal to the area of the parallelogram spanned by the vectors $\boldsymbol{u}$ and $a\boldsymbol{v}$, hence:

$$a(\boldsymbol{u} \wedge \boldsymbol{v}) = (a\boldsymbol{u}) \wedge \boldsymbol{v} = \boldsymbol{u} \wedge (a\boldsymbol{v})$$

- there is also a distributive law:

$$\boldsymbol{u} \wedge \boldsymbol{w} + \boldsymbol{v} \wedge \boldsymbol{w} = (\boldsymbol{u} + \boldsymbol{v}) \wedge \boldsymbol{w},$$

which can be visualised by the following picture:



(Notice that the cross product $\boldsymbol{u} \times \boldsymbol{v}$ satisfies the same properties.)

Differential 2-forms can now be understood in the same way as bivectors, with the difference that differential 2-forms are built from differential 1-forms instead of vectors. Hence, differential 2-forms on $\mathbb{R}^3$ are expressions of the form

$$F_1(x,y,z)\,\mathrm{d}x \wedge\, \mathrm{d}y + F_2(x,y,z)\,\mathrm{d}y \wedge\, \mathrm{d}z + F_3(x,y,z)\,\mathrm{d}z \wedge\, \mathrm{d}x.$$

Let $\mathrm{d}x$, $\mathrm{d}y$ and $\mathrm{d}z$ be differential 1-forms and let f be a differential 0-form. Then the following rules apply:

1. $\mathrm{d}x \wedge \mathrm{d}y = -\mathrm{d}y \wedge \mathrm{d}x$
2. $\mathrm{d}x \wedge \mathrm{d}x = 0$, where 0 stands for the nullform $0\,\mathrm{d}x + 0\,\mathrm{d}y + 0\,\mathrm{d}z$.
3. $f(\mathrm{d}x \wedge \mathrm{d}y) = (f\,\mathrm{d}x) \wedge \mathrm{d}y = \mathrm{d}x \wedge (f\,\mathrm{d}y)$
4. $(\mathrm{d}x \wedge \mathrm{d}z) + (\mathrm{d}y \wedge \mathrm{d}z) = (\mathrm{d}x + \mathrm{d}y) \wedge \mathrm{d}z$

Example
Let $\omega_1 = (x^2+y)\,\mathrm{d}x + (z - \sin y)\,\mathrm{d}y + xyz\,\mathrm{d}z$ and $\omega_2 = xz\,\mathrm{d}x + \cos y\,\mathrm{d}y$ be differential 1-forms. Then, using the rules for $\wedge$ we obtain

$$\begin{aligned}
\omega_1 \wedge \omega_2 &= ((x^2+y)\,\mathrm{d}x + (z-\sin y)\,\mathrm{d}y + xyz\,\mathrm{d}z) \wedge (xz\,\mathrm{d}x + \cos y\,\mathrm{d}y) \\
&= (x^2+y)xz\,\mathrm{d}x \wedge \mathrm{d}x + (x^2+y)\cos y\,\mathrm{d}x \wedge \mathrm{d}y \\
&\quad + (z-\sin y)xz\,\mathrm{d}y \wedge \mathrm{d}x + (z-\sin y)\cos y\,\mathrm{d}y \wedge \mathrm{d}y \\
&\quad + x^2yz^2\,\mathrm{d}z \wedge \mathrm{d}x + xyz\cos y\,\mathrm{d}z \wedge \mathrm{d}y \\
&= [(x^2+y)\cos y - (z-\sin y)xz]\,\mathrm{d}x \wedge \mathrm{d}y \\
&\quad - xyz\,\mathrm{d}y \wedge \mathrm{d}z - x^2yz^2\,\mathrm{d}z \wedge \mathrm{d}x.
\end{aligned}$$

□

Again, we can associate a vector field $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ to differential 2-form. We obtain this association in the following way. First let us use the associations of differential 1-forms with vectors as above:

$$\mathrm{d}x \leftrightarrow \widehat{\boldsymbol{i}}, \quad \mathrm{d}y \leftrightarrow \widehat{\boldsymbol{j}}, \quad \mathrm{d}z \leftrightarrow \widehat{\boldsymbol{k}}.$$

Then the wedge products of the differential 1-forms are associated with the cross products of the vectors, that is

$$\mathrm{d}x \wedge \mathrm{d}y \leftrightarrow \widehat{\boldsymbol{i}} \times \widehat{\boldsymbol{j}} = \widehat{\boldsymbol{k}}, \quad \mathrm{d}y \wedge \mathrm{d}z \leftrightarrow \widehat{\boldsymbol{j}} \times \widehat{\boldsymbol{k}} = \widehat{\boldsymbol{i}}, \quad \mathrm{d}x \wedge \mathrm{d}z \leftrightarrow \widehat{\boldsymbol{i}} \times \widehat{\boldsymbol{k}} = -\widehat{\boldsymbol{j}}.$$

By changing the order in the last wedge product we see that $\mathrm{d}z \wedge \mathrm{d}x$ is associated with $\widehat{\boldsymbol{j}}$. Hence, we associate a vector field over $\mathbb{R}^3$ with a differential 2-form in the following way:

$$F_1\widehat{\boldsymbol{i}} + F_2\widehat{\boldsymbol{j}} + F_3\widehat{\boldsymbol{k}} \leftrightarrow F_1\,\mathrm{d}y \wedge \mathrm{d}z + F_2\,\mathrm{d}z \wedge \mathrm{d}x + F_3\,\mathrm{d}x \wedge \mathrm{d}y.$$

The Hodge star operator maps differential 1-forms to differential 2-forms and back in the following way:

$$\star(F_1\,\mathrm{d}x + F_2\,\mathrm{d}y + F_3\,\mathrm{d}z) \leftrightarrow F_1\,\mathrm{d}y \wedge \mathrm{d}z + F_2\,\mathrm{d}z \wedge \mathrm{d}x + F_3\,\mathrm{d}x \wedge \mathrm{d}y$$

$$\star(F_1\,\mathrm{d}y \wedge \mathrm{d}z + F_2\,\mathrm{d}z \wedge \mathrm{d}x + F_3\,\mathrm{d}x \wedge \mathrm{d}y) \leftrightarrow F_1\,\mathrm{d}x + F_2\,\mathrm{d}y + F_3\,\mathrm{d}z$$

Surface integrals

We consider now surface integrals and show how they can be written using differential 2-forms. Let us consider a special case first where the surface is a suitable region D in $\mathbb{R}^2$. Let $\omega = F\,\mathrm{d}x \wedge \mathrm{d}y$ be a differential 2 form where $F : D \to \mathbb{R}$. Then we define the integral of the differential 2-form ω by

$$\int_D \omega = \iint_D F\,\mathrm{d}x \wedge \mathrm{d}y = \iint_D F\,\mathrm{d}x\,\mathrm{d}y.$$

(Notice that we write $\int_D \omega$ rather than $\iint_D \omega$ since, in general, we can have differential k-forms for which we would need k integral signs. Since in the general case this becomes too cumbersome one just writes only one integral sign.)

Assume that now the surface $S \subset \mathbb{R}^3$ is parameterised by $\Phi(u,v) = (x(u,v), y(u,v), z(u,v))$, where D is the domain of u, v. Note that

$$\begin{aligned}
\mathrm{d}y \wedge \mathrm{d}z &= \left(\tfrac{\partial y}{\partial u}\,\mathrm{d}u + \tfrac{\partial y}{\partial v}\,\mathrm{d}v\right) \wedge \left(\tfrac{\partial z}{\partial u}\,\mathrm{d}u + \tfrac{\partial z}{\partial v}\,\mathrm{d}v\right) = \tfrac{\partial(y,z)}{\partial(u,v)}\,\mathrm{d}u \wedge \mathrm{d}v \\
\mathrm{d}z \wedge \mathrm{d}x &= \left(\tfrac{\partial z}{\partial u}\,\mathrm{d}u + \tfrac{\partial z}{\partial v}\,\mathrm{d}v\right) \wedge \left(\tfrac{\partial x}{\partial u}\,\mathrm{d}u + \tfrac{\partial x}{\partial v}\,\mathrm{d}v\right) = \tfrac{\partial(z,x)}{\partial(u,v)}\,\mathrm{d}u \wedge \mathrm{d}v \\
\mathrm{d}x \wedge \mathrm{d}y &= \left(\tfrac{\partial x}{\partial u}\,\mathrm{d}u + \tfrac{\partial x}{\partial v}\,\mathrm{d}v\right) \wedge \left(\tfrac{\partial y}{\partial u}\,\mathrm{d}u + \tfrac{\partial y}{\partial v}\,\mathrm{d}v\right) = \tfrac{\partial(x,y)}{\partial(u,v)}\,\mathrm{d}u \wedge \mathrm{d}v
\end{aligned}$$

See Chapter 4, Section 1 for the definition of $\frac{\partial(x,y)}{\partial(u,v)}$.

Exercise
Verify the properties 1., 2., 3., 4. of the wedge product defined above for $x = x(u,v)$, $y = y(u,v)$ and $z = z(u,v)$. □

Let $F_1, F_2, F_3 : \mathbb{S} \to \mathbb{R}$. Then we have

$$\begin{aligned}
\iint_S F_1\,\mathrm{d}y \wedge \mathrm{d}z &= \iint_D F_1(x(u,v), y(u,v), z(u,v)) \tfrac{\partial(y,z)}{\partial(u,v)}\,\mathrm{d}u \wedge \mathrm{d}v \\
&= \iint_D F_1(x(u,v), y(u,v), z(u,v)) \tfrac{\partial(y,z)}{\partial(u,v)}\,\mathrm{d}u\,\mathrm{d}v, \\
\iint_S F_2\,\mathrm{d}z \wedge \mathrm{d}x &= \iint_D F_2(x(u,v), y(u,v), z(u,v)) \tfrac{\partial(z,x)}{\partial(u,v)}\,\mathrm{d}u\,\mathrm{d}v, \\
\iint_S F_3\,\mathrm{d}x \wedge \mathrm{d}y &= \iint_D F_3(x(u,v), y(u,v), z(u,v)) \tfrac{\partial(x,y)}{\partial(u,v)}\,\mathrm{d}u\,\mathrm{d}v.
\end{aligned}$$

Let a differential 2-form $\omega = F_1\,\mathrm{d}y \wedge \mathrm{d}z + F_2\,\mathrm{d}z \wedge \mathrm{d}x + F_3\,\mathrm{d}x \wedge \mathrm{d}y$ be given. Let $\boldsymbol{F} = F_1\widehat{\boldsymbol{i}} + F_2\widehat{\boldsymbol{j}} + F_3\widehat{\boldsymbol{k}}$ be the associated vector field. Then, by the definition above, we have

$$\int_S \omega = \iint_S \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S}.$$

To show the result see Chapter 4, Section 3.

Notice that the last equation also shows that the ordering of the sum in $\omega = F_1 \, \mathrm{d}y \wedge \mathrm{d}z + F_2 \, \mathrm{d}z \wedge \mathrm{d}x + F_3 \, \mathrm{d}x \wedge \mathrm{d}y$ is the most natural form.

Differential forms

To define general differential forms, we introduce one more rule. Let $\mathrm{d}x$, $\mathrm{d}y$ and $\mathrm{d}z$ be differential forms.

- Distributive law: $(\mathrm{d}x \wedge \mathrm{d}y) \wedge \mathrm{d}z = \mathrm{d}x \wedge (\mathrm{d}y \wedge \mathrm{d}z)$

Differential 3-forms over $\mathbb{R}^3$ are expressions of the form

$$F(x,y,z) \, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z,$$

where $F: \mathbb{R}^3 \to \mathbb{R}$. In general, differential k-forms defined over $\mathbb{R}^n$ are expressions of the form

$$\omega = F_1 \, \mathrm{d}x_{i_{1,1}} \wedge \ldots \wedge \mathrm{d}x_{i_{k,1}} + \cdots + F_r \, \mathrm{d}x_{i_{1,r}} \wedge \mathrm{d}x_{i_{k,r}},$$

where $F_i : \mathbb{R}^n \to \mathbb{R}$.

(Note that the cross product of vectors does not generalise to arbitrary dimensions, see here. Hence we do not use the analogy with the cross product of vectors anymore. Instead, one uses alternating multilinear forms.)

We call a differential form ω continuously differentiable if $F_1, \ldots, F_r$ are continuously differentiable (analogously we define twice continuously differentiable differential forms, and so on).

Some remarks:

- For each $k \ge 0$ there is a zero differential k-form 0 such that $0 + \omega = \omega$ and $0 \wedge \eta = 0$ for any differential k-form ω and any differential ℓ-form η. We call this form the nullform.
- A differential 0-form f is different from the nullform 0. For each $k \ge 0$, the nullform $0 = \sum_s 0 \, \mathrm{d}x_{i_{1,s}} \wedge \ldots \wedge \mathrm{d}x_{i_{k,s}}$, where the sum is over all subsets $\{i_{1,s}, \ldots, i_{k,s}\}$ of $\{1, \ldots, n\}$ consisting of k elements, is a differential k-form.

  In particular, for $\mathbb{R}^3$ we have the nullforms $f(\boldsymbol{x}) = 0$ for all $\boldsymbol{x} \in \mathbb{R}^3$ which is a differential 0-form, the nullform $0 \, \mathrm{d}x + 0 \, \mathrm{d}y + 0 \, \mathrm{d}z$ which is a differential 1-form, the nullform $0 = 0 \, \mathrm{d}x \wedge \mathrm{d}y + 0 \, \mathrm{d}y \wedge \mathrm{d}z + 0 \, \mathrm{d}z \wedge \mathrm{d}x$ which is a differential 2-form and the nullform $0 = 0 \, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ which is a differential 3-form.

- When adding differential forms ω_1 and ω_2, then both must be differential k-forms defined on $\mathbb{R}^n$ (or a subset of $\mathbb{R}^n$) for some $k \ge 0$ and $n \ge 1$. For instance, the expression $\mathrm{d}x + \mathrm{d}x \wedge \mathrm{d}y$ is not permitted (does not make sense).
- On the other hand the expression $\omega_1 \wedge \omega_2$, where ω_1 is a differential k-form and ω_2 is a differential ℓ-form, both defined over $\mathbb{R}^n$ (or an open subset of $\mathbb{R}^n$), is well defined. For instance, we have

$$\mathrm{d}x \wedge (x^2 \mathrm{d}y \wedge \mathrm{d}z) = x^2 \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z.$$

- Let ω_1 be a differential k-form and ω_2 be a differential ℓ-form, both defined over $\mathbb{R}^n$. If $k + \ell > n$, then the differential $k + \ell$-form $\omega_1 \wedge \omega_2 = 0$ is the nullform. The proof of this result is left as an exercise.

Let ω, η be differential forms. Then the wedge product satisfies the following properties.

1. For each $k \geq 0$ there exists a nullform 0 which is a differential k-form. The nullform satisfies $0 + \omega = \omega$ for all differential k-forms 0 and ω and $0 \wedge \eta = 0$ for all differential ℓ-forms η;
2. For any differential 0-form f and any differential k-form ω we have $f \wedge \omega = f\omega$;
3. $(f\omega_1 + \omega_2) \wedge \eta = f(\omega_1 \wedge \eta) + (\omega_2 \wedge \eta)$;
4. $\omega \wedge \eta = (-1)^{k\ell}(\eta \wedge \omega)$;
5. For differential k_i-forms ω_i, $i = 1, 2, 3$, we have $\omega_1 \wedge (\omega_2 \wedge \omega_3) = (\omega_1 \wedge \omega_2) \wedge \omega_3$;
6. Let f be a differential 0-form. Then $\omega \wedge (f\eta) = (f\omega) \wedge \eta = f(\omega \wedge \eta)$;

Exterior Derivative

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a differential 0-form. Then we have already seen that the derivative of this differential form is the differential 1-form $\mathrm{d}f = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y + \frac{\partial f}{\partial z}\,\mathrm{d}z$.

If ω is a differential k-form, then the derivative of ω, denoted by $\mathrm{d}\omega$, is a differential $(k+1)$-form. The operator d is called the exterior derivative.

The exterior derivative d satisfies the following properties:

- Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable differential 0-form. Then

$$\mathrm{d}f = \frac{\partial f}{\partial x_1}\,\mathrm{d}x_1 + \cdots + \frac{\partial f}{\partial x_n}\,\mathrm{d}x_n.$$

- If ω_1 and ω_2 are differential k-forms, then

$$\mathrm{d}(\omega_1 + \omega_2) = \mathrm{d}\omega_1 + \mathrm{d}\omega_2.$$

- For a continuously differentiable 0-form f and a differential k-form $\omega = f\,\mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_k}$, where $i_1, \ldots, i_k \in \{1, \ldots, n\}$, we define

$$\mathrm{d}\omega = \mathrm{d}f \wedge \mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_k}.$$

Laws for the exterior derivative
Let ω, η be continuously differentiable differential forms.

- $\mathrm{d}(\omega + \eta) = \mathrm{d}\omega + \mathrm{d}\eta$, if ω and η are both differential k-forms;
- $\mathrm{d}(c\omega) = c\,\mathrm{d}\omega$ for all $c \in \mathbb{R}$;
- $\mathrm{d}(\omega \wedge \eta) = \mathrm{d}\omega \wedge \eta + (-1)^k \omega \wedge \mathrm{d}\eta$, where ω is a differential k-form;

See also the de Rham cohomology.

Three Theorems

We have already seen how the fundamental theorem of line integrals can be written in a concise form using differentials. We now consider Green’s theorem, the Divergence theorem and Stokes’ theorem.

1. ***Green’s theorem***

   We now show how the formula in Green’s theorem can be obtained using differentials. Let $F_1, F_2 : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable and let $\omega = F_1\,\mathrm{d}x + F_2\,\mathrm{d}y$. Then

$$\begin{aligned}
\mathrm{d}\omega &= \mathrm{d}F_1 \wedge \mathrm{d}x + \mathrm{d}F_2 \wedge \mathrm{d}y \\
&= \left(\frac{\partial F_1}{\partial x}\,\mathrm{d}x + \frac{\partial F_1}{\partial y}\,\mathrm{d}y\right) \wedge \mathrm{d}x + \left(\frac{\partial F_2}{\partial x}\,\mathrm{d}x + \frac{\partial F_2}{\partial y}\,\mathrm{d}y\right) \wedge \mathrm{d}y \\
&= \frac{\partial F_1}{\partial y}\,\mathrm{d}y \wedge \mathrm{d}x + \frac{\partial F_2}{\partial x}\,\mathrm{d}x \wedge \mathrm{d}y \\
&= \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\,\mathrm{d}x \wedge \mathrm{d}y.
\end{aligned}$$

Let $\Omega \subset \mathbb{R}^2$ be a suitable domain (see here for more information) and let $\partial\Omega$ denote the boundary curve oriented counterclockwise. Let ω be a continuously differentiable differential 1-form defined on Ω. Then Green's theorem states that

$$\int_{\partial\Omega} \omega = \int_\Omega \mathrm{d}\omega.$$

2. ***Divergence theorem***

Let the functions $F_1, F_2, F_3 : \Omega \subset \mathbb{R}^3 \to \mathbb{R}$ be continuously differentiable and let $\omega = F_1 \, \mathrm{d}y \wedge \mathrm{d}z + F_2 \, \mathrm{d}z \wedge \mathrm{d}x + F_3 \, \mathrm{d}x \wedge \mathrm{d}y$. Then

$$\begin{aligned} \mathrm{d}\omega &= \mathrm{d}F_1 \wedge \mathrm{d}y \wedge \mathrm{d}z + \mathrm{d}F_2 \wedge \mathrm{d}z \wedge \mathrm{d}x + \mathrm{d}F_3 \wedge \mathrm{d}x \wedge \mathrm{d}y \\ &= \tfrac{\partial F_1}{\partial x} \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z + \tfrac{\partial F_2}{\partial y} \mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}x + \tfrac{\partial F_3}{\partial z} \mathrm{d}z \wedge \mathrm{d}x \wedge \mathrm{d}y \\ &= \left(\tfrac{\partial F_1}{\partial x} + \tfrac{\partial F_2}{\partial y} + \tfrac{\partial F_3}{\partial z} \right) \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \\ &= \mathrm{div}\, \boldsymbol{F} \, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z. \end{aligned}$$

Let $\partial\Omega$ be a closed, bounded and smooth surface and let Ω be the region enclosed by Ω. Then we can write the divergence theorem as

$$\int_{\partial\Omega} \omega = \int_\Omega \mathrm{d}\omega.$$

3. ***Stokes' Theorem***

Let the functions $F_1, F_2, F_3 : \Omega \subset \mathbb{R}^3 \to \mathbb{R}$ be continuously differentiable and let $\omega = F_1 \, \mathrm{d}x + F_2 \, \mathrm{d}y + F_3 \, \mathrm{d}z$. Then

$$\begin{aligned} \mathrm{d}\omega &= \mathrm{d}F_1 \wedge \mathrm{d}x + \mathrm{d}F_2 \wedge \mathrm{d}y + \mathrm{d}F_3 \wedge \mathrm{d}z \\ &= \left(\tfrac{\partial F_1}{\partial x} \mathrm{d}x + \tfrac{\partial F_1}{\partial y} \mathrm{d}y + \tfrac{\partial F_1}{\partial z} \mathrm{d}z \right) \wedge \mathrm{d}x \\ &\quad + \left(\tfrac{\partial F_2}{\partial x} \mathrm{d}x + \tfrac{\partial F_2}{\partial y} \mathrm{d}y + \tfrac{\partial F_2}{\partial z} \mathrm{d}z \right) \wedge \mathrm{d}y \\ &\quad + \left(\tfrac{\partial F_3}{\partial x} \mathrm{d}x + \tfrac{\partial F_3}{\partial y} \mathrm{d}y + \tfrac{\partial F_3}{\partial z} \mathrm{d}z \right) \wedge \mathrm{d}z \\ &= \left(\tfrac{\partial F_3}{\partial y} - \tfrac{\partial F_2}{\partial z} \right) \mathrm{d}y \wedge \mathrm{d}z + \left(\tfrac{\partial F_1}{\partial z} - \tfrac{\partial F_3}{\partial x} \right) \mathrm{d}z \wedge \mathrm{d}x \\ &\quad + \left(\tfrac{\partial F_2}{\partial x} - \tfrac{\partial F_1}{\partial y} \right) \mathrm{d}x \wedge \mathrm{d}y. \end{aligned}$$

Let Ω be a smooth and bounded surface and let $\partial\Omega$ be its boundary which we assume to be oriented positively. We have

$$\int_\Omega \mathrm{d}\omega = \iint_\Omega \mathrm{curl}\ \boldsymbol{F} \, \mathrm{d}\boldsymbol{S}.$$

Hence we can write Stokes' theorem as

$$\int_{\partial\Omega} \omega = \int_\Omega \mathrm{d}\omega.$$

Stokes' theorem

We have now seen that all the main theorems can be written in a single form. It now becomes obvious how to generalise the results above to arbitrary dimension and manifolds. This theorem is called Stokes' theorem (named after the same person as the theorem considered in Chapter 5, Section 2).

> ***Stokes' theorem***
> Let Ω be an oriented smooth manifold of dimension n. Let ω be a differential $(n-1)$-form with compact support on Ω and let $\partial\Omega$ denote the boundary of Ω with its induced orientation, then
>
> $$\int_{\partial\Omega} \omega = \int_{\Omega} \mathrm{d}\omega.$$

Exercise
Let $\Omega \subset \mathbb{R}^4$ be an oriented smooth manifold of dimension 3. Let $\omega = F_1 \,\mathrm{d}x_1 \wedge \mathrm{d}x_2 + F_2 \,\mathrm{d}x_1 \wedge \mathrm{d}x_3 + F_3 \,\mathrm{d}x_1 \wedge \mathrm{d}x_4 + F_4 \,\mathrm{d}x_2 \wedge \mathrm{d}x_3 + F_5 \,\mathrm{d}x_2 \wedge \mathrm{d}x_4 + F_6 \,\mathrm{d}x_3 \wedge \mathrm{d}x_4$ be a differential 2-form.
Write down Stokes' theorem in explicit form.
□

Closed and exact differential forms

> ***Definition***
> A differential form ω for which $\mathrm{d}\omega = 0$ (the nullform) is called **closed**.

Exercise
Let $F_1, F_2, F_3 : \mathbb{R}^3 \to \mathbb{R}$ be twice continuously differentiable. Define the vector field $\boldsymbol{F} = F_1\widehat{\boldsymbol{i}} + F_2\widehat{\boldsymbol{j}} + F_3\widehat{\boldsymbol{k}}$ and the associated differential 1-form $\omega = F_1 \,\mathrm{d}x + F_2 \,\mathrm{d}y + F_3 \,\mathrm{d}z$. Show that $\mathrm{curl}\ \boldsymbol{F} = \boldsymbol{0}$ if and only if ω is closed. □

We have previously defined conservative vector fields $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$, see Chapter 3, Section 3. A similar concept applies to differentials.

> ***Definition***
> A differential form ω is called **exact** if there is a differential form η such that $\omega = \mathrm{d}\eta$.

Exercise
Let $F_1, F_2, F_3 : \mathbb{R}^3 \to \mathbb{R}$ be continuous. Define the vector field $\boldsymbol{F} = F_1\widehat{\boldsymbol{i}} + F_2\widehat{\boldsymbol{j}} + F_3\widehat{\boldsymbol{k}}$ and the associated differential 1-form $\omega = F_1 \,\mathrm{d}x + F_2 \,\mathrm{d}y + F_3 \,\mathrm{d}z$. Show that $\boldsymbol{F}$ is conservative if and only if ω is exact. □

> ***Poincaré lemma***
> If Ω is a contractible open subset of $\mathbb{R}^n$, any smooth closed differential k-form ω defined on Ω is exact, for any integer $k>0$ (this has content only when $k \le n$).

In Chapter 2, Section 2 we have seen that $\mathrm{div}\ \mathrm{curl}\ \boldsymbol{F} = 0$ and $\mathrm{curl}\ \mathrm{grad}\ \boldsymbol{F} = \boldsymbol{0}$. For differential forms we have the following result.

> ***Proposition***
> Let ω be a twice continuously differentiable form defined on an open set of $\Omega \subseteq \mathbb{R}^n$. Then
>
> $$\mathrm{d}(\mathrm{d}\omega) = 0.$$

The proof is left as exercise. (*Hint* Use Clairaut's theorem.)

Notice that there is an analogue to this proposition for the regions of integration. Namely, let $\Omega \subset \mathbb{R}^n$ be compact and assume that Ω has a smooth boundary. Then the boundary of the boundary of Ω is empty, that is,

$$\partial(\partial\Omega) = \emptyset.$$

Cauchy's integral theorem

Cauchy's integral theorem is an important result in complex analysis. It can also be expressed using differential forms (since it is somewhat related to Green's theorem.) Let $\mathbb{C}$ denote the set of complex numbers.

Let $z = x + \mathrm{i}y \in \mathbb{C}$. A complex function $f : \mathbb{C} \to \mathbb{C}$ can be written as

$$f(z) = u(x, y) + \mathrm{i}v(x, y),$$

where $u, v : \mathbb{R}^2 \to \mathbb{R}$. It can be shown that the function f is continuous if both u and v are continuous. We now consider complex derivatives. We set

$$f'(z) = \lim_{h \to 0} \frac{f(z + h) - f(h)}{h},$$

where h is a complex number. Hence for the derivative to be well defined we need to demand that the limit is the same for each way h approaches 0. In particular, if we restrict h to real numbers, we get

$$\lim_{h \in \mathbb{R}, h \to 0} \frac{u(x + h, y) - u(x, y)}{h} + \mathrm{i}\frac{v(x + h, y) - v(x, y)}{h} = \frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x}.$$

On the other hand, if we restrict h to imaginary numbers, we get

$$\lim_{h = a\mathrm{i}, a \in \mathbb{R}, h \to 0} \frac{u(x, y + a) - u(x, y)}{a\mathrm{i}} + \frac{v(x, y + a) - v(x, y)}{a} = -\mathrm{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

Since the limits must coincide we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

These equations are called the Cauchy-Riemann equations. The converse also holds, that is, if the u, v are differentiable and the Cauchy-Riemann equations hold, then f has a complex derivative. If f has a complex derivative at some point $z \in \mathbb{C}$, then we say that f is analytic at z.

A complex differential form ω is an expression of the form $\omega = \eta + \mathrm{i}\mu$, where η and μ are differential k-forms for some $k \geq 0$. We define the integral of a complex differential 1 form by

$$\int_C \omega = \int_C \eta + \mathrm{i} \int_C \mu,$$

where C can be parameterised by $\mathbf{c}(t) = \alpha(t) + \mathrm{i}\beta(t)$ where $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ are piecewise continuously differentiable.

Let $f = u + \mathrm{i}v$ and $\mathrm{d}z = \mathrm{d}x + \mathrm{i}\,\mathrm{d}y$. Then

$$f(z)\,\mathrm{d}z = (u + \mathrm{i}v)(\mathrm{d}x + \mathrm{i}\,\mathrm{d}y) = u\,\mathrm{d}x - v\mathrm{d}y + \mathrm{i}(v\,\mathrm{d}x + u\,\mathrm{d}y).$$

If f is analytic, then we obtain

$$\mathrm{d}(f(z)\,\mathrm{d}z) = -\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\,\mathrm{d}x \wedge \mathrm{d}y + \mathrm{i}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\,\mathrm{d}x \wedge \mathrm{d}y = 0.$$

Hence, by Green's theorem, we obtain the following result.

> ***Cauchy's integral theorem***
> Let C be a simple closed curve parameterised by a piecewise continuously differentiable function $\mathbf{c}$. Then for any analytic function $f : D \to \mathbb{C}$, where $D \subset \mathbb{C}$ is the region enclosed by C we have

$$\int_C f(z)\,\mathrm{d}z = 0.$$

Exercise
Prove Cauchy's integral theorem using Green's theorem (without the use of differentials).□

More information

For more information on differentials see here and here.

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