Quasi-Random Ideas. By Josef Dick.

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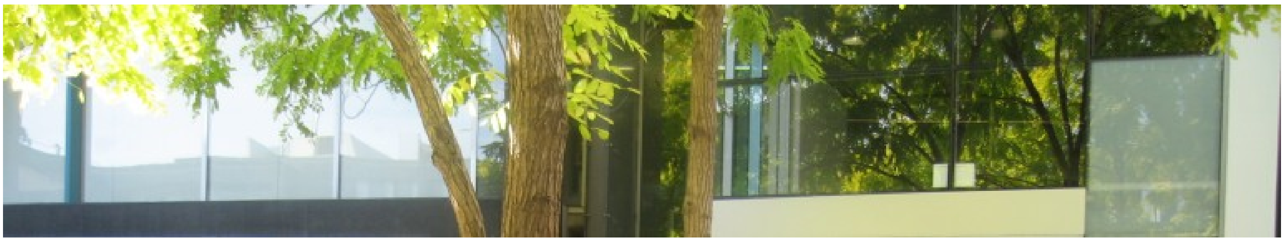

← Math2111: Chapter 3: Line integrals. Section 1: Scalar line integrals

Math2111: Chapter 3: Line integrals. Section 2: Vector line integrals

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In this blog entry you can find lecture notes for Math2111, several variable calculus. See also the table of contents for this course. This blog entry printed to pdf is available here.

Work integral

Previously we considered integrals of a function $f:\mathbb{R}^n \to \mathbb{R}$ (n=2,3) over a curve $\mathcal{C}$. We now turn to integrating vector fields $\mathbb{F}:\mathbb{R}^n \to \mathbb{R}^n$ where $n=2$ or 3. These integrals can be motivated by calculating the work done by a force on a particle along some curve $\mathcal{C}$.

In the simplest case, the work W done a force $\mathbb{F}$ acting on an object which moves along a straight line is given by

$$W = \mathbb{F} \cdot \frac{\vec{d}}{\|\vec{d}\|} \|\vec{d}\| = \mathbb{F}\cdot \vec{d},$$

where the object is displaced in the direction of the vector $\vec{d}$ with distance $\|\vec{d}\|$.

In a more general setting, the object moves along a curve $\boldsymbol{r}:[a,b]\to\mathbb{R}^n$. One way to arrive at a formula for calculating the work done by a force $\boldsymbol{F}$ along $\boldsymbol{r}$ is to take the component of $\boldsymbol{F}$ in the tangential direction on each point on the curve and integrate this quantity using a scalar line integral. The tangential direction on the curve at a point $\boldsymbol{r}(t)$ is given by $\boldsymbol{r}'(t)$ and a unit vector in tangential direction is given by

$$\widehat{\boldsymbol{T}}(t) = \frac{\boldsymbol{r}'(t)}{\|\boldsymbol{r}'(t)\|},$$

where $\|\cdot\|$ denotes the Euclidian norm and where we assume that $\|\boldsymbol{r}'(t)\| \neq 0$ for all $t \in [a,b]$. Hence the component of $\boldsymbol{F}$ in the direction of the tangent to the curve at a point $\boldsymbol{r}(t)$ is given by

$$\boldsymbol{F}(\boldsymbol{r}(t)) \cdot \widehat{\boldsymbol{T}}(t).$$

This is now a scalar valued function which can be integrated using the scalar line integral. We thus obtain that the work done is given by

$$W = \int_a^b \boldsymbol{F}(\boldsymbol{r}) \cdot \widehat{\boldsymbol{T}} \, \mathrm{d}s = \int_a^b \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t) \, \mathrm{d}t,$$

where we used $\|\boldsymbol{r}'(t)\| \, \mathrm{d}t = \mathrm{d}s$.

Line integral

We can now formally define line integrals of vector fields over some curve.

> ***Line Integral***
> Let $n = 2$ or 3. Let $\boldsymbol{r} : [a,b] \to \mathbb{R}^n$ be a continuously differentiable curve and let $\boldsymbol{F} : D \to \mathbb{R}^n$ be a continuous vector field, where we assume that $\{\boldsymbol{r}(t) : t \in [a,b]\} \subseteq D$. Then we define $\int_c \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s}$, the line integral of $\boldsymbol{F}$ along $\boldsymbol{r}$, by the formula
>
> $$\int_{\boldsymbol{r}} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = \int_a^b \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t) \, \mathrm{d}t.$$

Notice that we do not need the assumption that $\boldsymbol{r}'(t) \neq 0$ for all $t \in [a,b]$. This can be shown by setting up Riemann sums as in the case for scalar line integrals. (This makes a difference in some cases. For example, for $\boldsymbol{r}(t) = (t^3, t^2)$ we get $\|\boldsymbol{r}'(0)\| = 0$, and this cannot be avoided using a different parameterisation.)

Line integrals are written in various forms. For instance, let $\boldsymbol{F} = (F_1, F_2, F_3)$ and $\boldsymbol{r}(t) = (x(t), y(t), z(t))$. Then the line integral is also written as

$$\int_{\boldsymbol{r}} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = \int_a^b \left(F_1 \frac{\mathrm{d}x}{\mathrm{d}t} + F_2 \frac{\mathrm{d}y}{\mathrm{d}t} + F_3 \frac{\mathrm{d}z}{\mathrm{d}t} \right) \mathrm{d}t = \int_a^b F_1 \, \mathrm{d}x + F_2 \, \mathrm{d}y + F_3 \, \mathrm{d}z.$$

There also exist analogous ways for writing line integrals for the two-dimensional case.

Example
Let $\boldsymbol{r}(t) = (t, t^2, t^3)$ for $0 \leq t \leq 1$ and $\boldsymbol{F}(x, y, z) = (\mathrm{e}^x, xy, xyz)$. Then from the parameterisation of the curve we have $x(t) = t$, $y(t) = t^2$ and $z(t) = t^3$. Hence $\mathrm{d}x = \mathrm{d}t$, $\mathrm{d}y = 2t \, \mathrm{d}t$ and $\mathrm{d}z = 3t^2 \, \mathrm{d}t$. Further $F_1(x(t), y(t), z(t)) = \mathrm{e}^t$, $F_2(x(t), y(t), z(t)) = t^3$, $F_3(x(t), y(t), z(t)) = t^6$. Hence

$$\int_{\boldsymbol{r}} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = \int_0^1 \mathrm{e}^t \, \mathrm{d}t + 2t^4 \, \mathrm{d}t + 3t^8 \, \mathrm{d}t = \mathrm{e} - \frac{4}{15}.$$

□

Some properties of line integrals

Let $\boldsymbol{F}, \boldsymbol{G}$ be continuous vector fields and let $k \in \mathbb{R}$ be a constant.

1. $\int_{\mathcal{C}} (\boldsymbol{F} + \boldsymbol{G}) \cdot \mathrm{d}\boldsymbol{s} = \int_{\mathcal{C}} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} + \int_{\mathcal{C}} \boldsymbol{G} \cdot \mathrm{d}\boldsymbol{s}$.
2. $\int_{\mathcal{C}} k\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = k \int_{\mathcal{C}} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s}$.
3. Let $\mathcal{C}$ be a smooth curve and let $-\mathcal{C}$ denote the same curve but with the orientation reversed. Then $\int_{-\mathcal{C}} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = -\int_{\mathcal{C}} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s}$.
4. Let $\mathcal{C}$ be a union of n smooth curves $\mathcal{C}_1, \ldots, \mathcal{C}_n$, then $\int_{\mathcal{C}} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = \int_{\mathcal{C}_1} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} + \cdots + \int_{\mathcal{C}_n} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s}$.

Line integrals in the plane in normal form

We defined line integrals by integrating the tangential component of a vector field $\boldsymbol{F}$ over a curve $\mathcal{C}$. In the plane it is also meaningful to compute the orthogonal component of the vector field along some curve $\mathcal{C}$. Let $\boldsymbol{r} : [a,b] \to \mathbb{R}^2$ with $\boldsymbol{r}(t) = (x(t), y(t))$ be a curve and let $\widehat{\boldsymbol{n}}(t)$

be a unit normal vector to the curve at the point $\boldsymbol{r}(t)$ which is obtained by turning the unit tangent vector $\widehat{\boldsymbol{T}}$ by 90° clockwise, then this line integral is given by

$$\int_C \boldsymbol{F} \cdot \widehat{\boldsymbol{n}} \,\mathrm{d}s = \int_a^b F_1 \,\mathrm{d}y - F_2 \,\mathrm{d}x = \int_a^b \boldsymbol{F} \cdot (y'(t), -x'(t)) \,\mathrm{d}t.$$

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