[Quasi-Random Ideas. By Josef Dick.]

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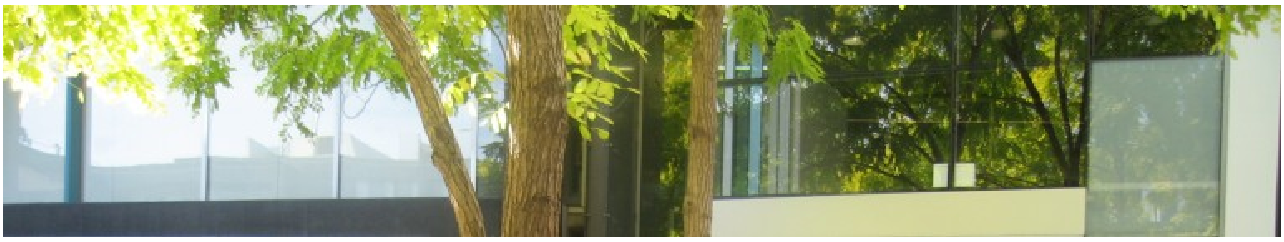

← Math2111: Chapter 3: Additional Material: For which curves and regions does Green's theorem apply?

Math2111: Chapter 3: Additional Material: Rectifiable parameterised curves

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In this blog entry you can find lecture notes for Math2111, several variable calculus. See also the table of contents for this course. This blog entry printed to pdf is available here.

We now discuss curves and their lengths in more detail. Let $n \in \mathbb{N}$ be an arbitrary natural number which is fixed throughout this post. Note that in general one needs to distinguish between a parameterised curve, which is a continuous mapping $\boldsymbol{c}: [a, b] \to \mathbb{R}^n$, and the curve $\mathcal{C}$, which is the image of $\boldsymbol{c}$ given by $\{\boldsymbol{c}(t) \in \mathbb{R}^n : t \in [a, b]\}$. Here we shall discuss parameterised curves. Hence, for instance, the parameterised curve $\boldsymbol{c}(t) = \cos t \widehat{\boldsymbol{\imath}} + \sin t \widehat{\boldsymbol{\jmath}}$ with $0 \le t \le 4\pi$ is a circle traversed twice and has therefore length 4π, whereas its image is just a circle which has length 2π.

Rectifiable parameterised curves

Let now $\boldsymbol{c}: [a, b] \to \mathbb{R}^n$ be a parameterised curve (note that this implies that $\boldsymbol{c}$ is continuous). We call a set $P_N = \{t_0, t_1, \ldots, t_N\}$ a partition of the interval $[a, b]$ (of length N) if $a = t_0 < t_1 < \cdots < t_{N-1} < t_N = b$. The set of all partitions of length N shall be denoted by $\mathcal{P}_N$ and the set of all partitions by $\mathcal{P} = \bigcup_{N=1}^\infty \mathcal{P}_N$.

We define

$$\ell(\boldsymbol{c}, P_N) = \sum_{n=1}^N \|\boldsymbol{c}(t_n) - \boldsymbol{c}(t_{n-1})\|_2,$$

where $\|\cdot\|_2$ denotes the Euclidean norm.

> ***Definition*** *(Rectifiable parameterised curve)*
> A parameterised curve $\boldsymbol{c}: [a, b] \to \mathbb{R}^n$ is called rectifiable if there is a constant $K > 0$ such that for all partitions $P_N \in \mathcal{P}$ of $[a, b]$ we have

$$\ell(\boldsymbol{c}, P_N) \leq K.$$

If $\boldsymbol{c}$ is rectifiable, then we define the length $\ell(\boldsymbol{c})$ of $\boldsymbol{c}$ by

$$\ell(\boldsymbol{c}) = \sup_{P_N \in \mathcal{P}} \ell(\boldsymbol{c}, P_N).$$

Example and Exercise
Assume that the parameterised curve $\boldsymbol{c} = (x_1, \ldots, x_n)$ is continuously differentiable, that is, each of its component functions x_k is continuously differentiable. Then $\boldsymbol{c}$ is rectifiable and

$$\ell(\boldsymbol{c}) = \int_a^b \|\boldsymbol{c}'(t)\|_2 \, \mathrm{d}t.$$

To show this observe that

$$\ell(\boldsymbol{c}, P_N) = \sum_{n=1}^N \|\boldsymbol{c}(t_n) - \boldsymbol{c}(t_{n-1})\|_2 = \sum_{n=1}^N \left\| \frac{\boldsymbol{c}(t_n) - \boldsymbol{c}(t_{n-1})}{t_n - t_{n-1}} \right\|_2 (t_n - t_{n-1}).$$

Now use the mean value theorem and obtain a Riemann sum. The details are left to the reader as an exercise. □

Variation

We now introduce the variation of a function.

Definition
Let $f : [a, b] \to \mathbb{R}$ be a function. Then the variation $V(f)$ of f is given by

$$V(f) = \sup_{P_N \in \mathcal{P}} \sum_{n=1}^N |f(t_n) - f(t_{n-1})|.$$

Exercise
Let $f : [a, b] \to \mathbb{R}$ be continuously differentiable. Show that $V(f) = \int_a^b |f'(t)| \, \mathrm{d}t$. □

If $V(f) < \infty$, then we say that f has bounded variation.

Functions of bounded variation have many useful properties. For example, if $V(f) < \infty$, then f is piecewise continuous and therefore Riemann integrable.

On the other hand, there are continuous functions which have unbounded variation.
Example and Exercise
For instance, the function

$$f(x) = \begin{cases} x \cos \frac{\pi}{x} & \text{for } x \in (0, 1], \\ 0 & \text{for } x = 0, \end{cases}$$

is continuous, but has unbounded variation. To show that it has unbounded variation consider the partitions $\{0, 1/(2N), 1/(2N-1), \ldots, 1/3, 1/2, 1\}$. The details are left as exercise. □

Rectifiable parameterised curves and bounded variation

We have now the following result.

Theorem
The parameterised curve $\boldsymbol{c} : [a, b] \to \mathbb{R}^n$ given by $\boldsymbol{c}(t) = (x_1(t), \ldots, x_n(t))$ is rectifiable if and only if each of the component functions x_k have bounded variation, i.e. $V(x_k) < \infty$ for $1 \leq k \leq n$.

For the proof of this result the following inequality is useful: for any real numbers $a_1, \ldots, a_n$

we have

$$\sqrt{a_1^2 + \cdots + a_n^2} \le |a_1| + \cdots + |a_n|.$$

This inequality can be shown by taking the square on each side. (It is also a special case of the important Jensen's inequality.) The details of the proof the theorem are left as an exercise.

Parameterised curves which are not rectifiable

Exercise
Find a parameterised curve in $\mathbb{R}^2$, that is, a continuous function $\boldsymbol{c} : [a, b] \to \mathbb{R}^2$, which is not rectifiable (i.e. has infinite length). (Hint: Use the function of unbounded variation defined above to define $\boldsymbol{c}$.) □

Remark
Consider the Koch curve. This curve is simple and closed and encloses a finite area, but the length of the curve is infinite. This shows that there are regions of finite area whose boundary has infinite length. □

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Tagged: bounded variation, curve, line integral, parameterised curve, rectifiable, variation

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