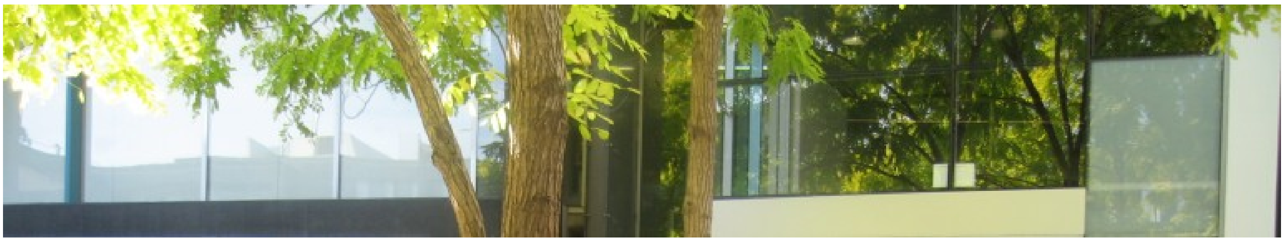
Quasi-Random Ideas. By Josef Dick.

- Front Page
- About
- Papers
- Writing LaTeX in WordPress
- Book
- Teaching



← Math2111: Chapter 1: Fourier series. Additional Material: L_2 convergence of Fourier series.

Math2111: Chapter 1: Fourier series. Section 6: Heat equation

April 22, 2010 · Leave a Comment · Edit This

In this blog entry you can find lecture notes for Math2111, several variable calculus. See also the table of contents for this course.

In this part we discuss applications of Fourier series to solving a certain type of partial differential equation (pde). In more detail, we discuss the heat equation. The aim is to show how Fourier series naturally come up in the solution of this equation.

The heat equation describes the heat distribution in space and time. To illustrate the method it is sufficient to consider only the case of one spatial variable here. Let $x \in [0, L]$ denote the spatial variable, let $t \in [0, \infty)$ denote time and let $u = u(x, t)$ be a function of x and t. Then the heat equation is given by

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2},$$

where $\alpha > 0$ is a constant. The task is to find a function $u : [0, L] \times [0, \infty) \to \mathbb{R}$ which satisfies the heat equation, some initial condition

$$u(x, 0) = f(x) \quad \text{for } x \in [0, L],$$

where $f : [0, L] \to \mathbb{R}$ is, say, continuous, and (in our case) homogeneous boundary conditions

$$u(0, t) = 0 \qquad \text{and } u(L, t) = 0.$$

Separation of variables

We solve the heat equation using separation of variables, that is, we assume that the solution to the problem is of the form

$$u(x, t) = X(x)T(t)$$

for a function $X:[0,L]\to\mathbb{R}$ and $T:[0,\infty)\to\mathbb{R}$ (note that X does NOT depend on t and that T does NOT depend on x). In this case we have

$$\frac{\partial u}{\partial t} = XT' \quad \text{and} \ \frac{\partial^2 u}{\partial x^2} = X''T,$$

where $T' = \frac{\mathrm{d}T}{\mathrm{d}t}$ and $X'' = \frac{\mathrm{d}^2 X}{\mathrm{d}x^2}$. Substituting this ansatz into the heat equation we obtain

$$XT' = \alpha^2 X'' T.$$

We can now separate the functions X and T to obtain

$$\frac{T'}{T} = \alpha^2 \frac{X''}{X}.$$

Now observe that

$$\frac{T'}{T}$$

is only a function of t, whereas

$$\frac{X''}{X}$$

is only a function of x. Hence, the only way those two expressions can be equal is if both

$$\frac{T'}{T} \quad \text{and} \ \frac{X''}{X}$$

are constant. Hence we have

$$\frac{T'}{T} = \alpha^2 \frac{X''}{X} = -\lambda^2.$$

Thus we obtain now two ordinary differential equations (ode)

$$X'' + p^2 X = 0, \qquad (1)$$

$$T' + \lambda^2 T = 0, \qquad (2)$$

where we set $p = \lambda/\alpha$. The first ode has a solution of the form

$$X(x) = A\cos px + B\sin px$$

(check this by substituting this equation into the ode). Now our solution should satisfy the boundary conditions

$$u(0,t) = X(0)T(t) = 0 \quad \text{and} \ u(L,t) = X(L)T(t) = 0.$$

If $X(0)\neq 0$ or if $X(L)\neq 0$, then $T(t)=0$ for all t and hence we only obtain the trivial solution $u(x,t)=0$ for all x and t. Hence we assume now that $X(0)=X(L)=0$, which implies that

$$X(0) = A\cos p0 = 0 \quad \text{and} \ X(L) = A\cos pL + B\sin pL = 0.$$

The first equation implies that $A=0$, hence we get from the second equation that

$$B\sin pL = 0.$$

If $B=0$, then $X(x)=0$ for all x and we only get the trivial solution. If $B\neq 0$, then $\sin pL = 0$. This holds if $pL = n\pi$ for some integer $n\geq 1$ (for $n=0$ we only get the trivial solution again). Indeed, we get infinitely many solutions, where $p\in\{\frac{\pi}{L}, \frac{2\pi}{L}, \frac{3\pi}{L}, \ldots\}$. Let now

$$p_n = \frac{n\pi}{L} \quad \text{for } n = 1, 2, 3, \ldots.$$

Then we obtain infinitely many solutions to the ode (1) of the form

$$X_n(x) = B_n \sin\left(\frac{n\pi x}{L}\right).$$

We now solve (2). We have $p_n = \lambda_n/\alpha$ (α is a constant given by the equation, but λ may take on different values depending on the solution). Hence $\lambda_n = \alpha p_n = \frac{n\alpha\pi}{L}$ for $n = 1, 2, 3, \ldots$. Hence for each natural number n we obtain a solution T_n to the ode (2) (where $\lambda = \lambda_n$) of the form

$$T_n(t) = \mathrm{e}^{-\lambda_n^2 t}$$

(check that this solves the ode (2)).

Thus we obtain solutions to the heat equation which satisfy the homogeneous boundary conditions of the form

$$u_n(x,t) = X_n(x) T_n(t) \quad \text{where } n = 1, 2, 3, \ldots.$$

We call these functions the **eigenfunctions** corresponding to the **eigenvalues** $\lambda_n = \frac{n\alpha\pi}{L}$.

We still need to find a solution which satisfies the initial condition. To do so, notice that any linear combination of the eigenfunctions u_n is again a solution to the heat equation which satisfies the homogeneous boundary conditions. Hence, in general, we have a solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \mathrm{e}^{-\lambda_n^2 t},$$

where $B_1, B_2, \ldots$ are real numbers which we can choose such that u satisfies the initial condition. More precisely, we need to choose $B_1, B_2, \ldots$ such that

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) = f(x) \quad \text{for all } x \in [0, L].$$

The last equation means that the B_n are the Fourier coefficients of the Fourier sine series of f. Hence

$$B_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) \, \mathrm{d}x \quad \text{for } n = 1, 2, 3, \ldots$$

which completes our solution to the heat equation.

Example Solve the heat equation in one dimension with homogeneous boundary conditions, assuming that the initial temperature is given by

$$f(x) = \begin{cases} x & \text{if } 0 < x < L/2, \\ L - x & \text{if } L/2 \leq x < L. \end{cases}$$

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  - About
  - Book
  - Papers
  - Teaching
    - Math2111: Several Variable Calculus: Table of Contents
  - Writing LaTeX in WordPress

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