# Equidistribution properties of generalized nets and sequences

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**Abstract** Generalized digital nets and sequences have been introduced for the numerical integration of smooth functions using quasi-Monte Carlo rules. In this paper we study geometrical properties of such nets and sequences. The definition of these nets and sequences does not depend on linear algebra over finite fields, it only requires that the point set or sequence satisfies certain distributional properties. Generalized digital nets and sequences appear as special cases. We prove some propagation rules and give bounds on the quality parameter $t$.

## 1 Introduction

In this paper we study the equidistribution properties of generalized digital nets and sequences as introduced in [2], see also [1, 3]. Such nets and sequences have been introduced since they can achieve arbitrarily high convergence rates of the integration error when used in a quasi-Monte Carlo rule as quadrature points. To be more precise, if the function $f : [0,1]^s \to \mathbb{R}$, $s \geq 1$, under consideration has mixed partial derivatives up to order $\alpha \geq 1$ in each variable which are square-integrable, then the integration error is of $\mathscr{O}(q^{-(\beta n - t)}(\beta n - t)^{s\alpha})$, for a digital $(t, \alpha, \beta, n \times m, s)$-net over $\mathbb{F}_q$. Explicit constructions of digital $(t, \alpha, \beta, n \times m, s)$-nets over $\mathbb{F}_q$ with $\beta n = \alpha m$ and $t$ bounded independently of $m$ are also given in [1, 2]. Note that a digital $(t, \alpha, \beta, n \times m, s)$-net over $\mathbb{F}_q$ has $q^m$ points.

In the next section we define digital $(t, \alpha, \beta, n \times m, s)$-nets and digital $(t, \alpha, \beta, \sigma, s)$-sequences and recall some of their properties as well as explicit constructions from

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[2]. In Section 3, generalized nets and sequences are introduced. In order to do so, we introduce the concept of a generalized elementary interval in Subsection 3.1. We prove some properties of such sets and then give the definition of $(t,\alpha,\beta,n,m,s)$-nets and $(t,\alpha,\beta,\sigma,s)$-sequences. In Subsection 3.2, propagation rules for these types of point sets and sequences are shown and we also prove some lower and upper bounds on the quality parameter $t$. In particular, we show that the quality parameter $t$ of a $(t,\alpha,\beta,\sigma,s)$-sequence with smallest possible value of $t$ satisfies $t \asymp \alpha^2 s$, which also holds for digital sequences. For the remainder of the paper we use the following nomenclature: $(t,\alpha,\beta,n,m,s)$-nets and $(t,\alpha,\beta,\sigma,s)$-sequences as introduced in Section 3 of this paper will be referred to as *generalized nets* and *generalized sequences*, digital $(t,\alpha,\beta,n\times m,s)$-nets and digital $(t,\alpha,\beta,\sigma,s)$-sequences, as introduced in [2], will be referred to as *generalized digital nets* and *generalized digital sequences*, $(t,m,s)$-nets and $(t,s)$-sequences, [9, 10] will be referred to as *classical nets* and *classical sequences*, digital $(t,m,s)$-nets and digital $(t,s)$-sequences, [9, 10] as *classical digital nets* and *classical digital sequences*.

## 2 Definition of digital $(t,\alpha,\beta,n\times m,s)$-nets and digital $(t,\alpha,\beta,\sigma,s)$-sequences

Before providing a geometric approach to digital $(t,\alpha,\beta,n\times m,s)$-nets, we need to recall the following concepts: We start with the digital construction scheme, which digital $(t,\alpha,\beta,n\times m,s)$-nets are based upon. This digital construction scheme stems from the construction of digital $(t,m,s)$-nets, see [10].

Throughout the paper $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{N}_0$ the set of nonnegative integers. Having defined digital $(t,\alpha,\beta,n\times m,s)$-nets and digital $(t,\alpha,\beta,\sigma,s)$-sequences, we will explain the meaning of the parameters in Remark 1.

**Definition 1.** Let $q$ be a prime power and let $n,m,s \geq 1$ be integers. Let $C_1,\ldots,C_s$ be $n\times m$ matrices over the finite field $\mathbb{F}_q$ of order $q$. Now we construct $q^m$ points in $[0,1)^s$: For $0 \leq h < q^m$ let $h = h_0 + h_1 q + \cdots + h_{m-1}q^{m-1}$ be the q-adic expansion of $h$. Consider an arbitrary but fixed bijection $\varphi : \{0,1,\ldots,q-1\} \to \mathbb{F}_q$. Identify $h$ with the vector $\mathbf{h} = (\varphi(h_0),\ldots,\varphi(h_{m-1}))^\top \in \mathbb{F}_q^m$, where $\top$ denotes the transpose of the vector. For $1 \leq j \leq s$, multiply the matrix $C_j$ by $\mathbf{h}$, i.e.,

$$C_j\mathbf{h} := (y_{j,1}(h),\ldots,y_{j,n}(h))^\top \in \mathbb{F}_q^n,$$

and set

$$x_{h,j} := \frac{\varphi^{-1}(y_{j,1}(h))}{q} + \cdots + \frac{\varphi^{-1}(y_{j,n}(h))}{q^n}.$$

The point set $\left\{x_0,\ldots,x_{q^m-1}\right\}$ is called a digital net (over $\mathbb{F}_q$) (with generating matrices $C_1,\ldots,C_s$). For $n,m = \infty$ we obtain a sequence $\{x_0,x_1,\ldots\}$, which is called a digital sequence (over $\mathbb{F}_q$) (with generating matrices $C_1,\ldots,C_s$).

It is clear from the definition, that all the information about the properties of the point set is contained in the generating matrices $C_1,\dots,C_s$. Hence in order to be able to deal with the properties of these point sets, it is enough to introduce a criterion on the generating matrices. To define such a criterion we first define the dual space [4, 5, 11] of the generating matrices $C_1,\dots,C_s \in \mathbb{F}_q^{n\times m}$ for a digital net, given by

$$\mathscr{D} = \{k \in \mathbb{N}_0^s : C_1^\top \mathbf{k}_1 + \cdots + C_s^\top \mathbf{k}_s = \mathbf{0} \in \mathbb{F}_q^m\},$$

where for $k = (k_1,\dots,k_s)$ with $k_j = k_{j,0} + k_{j,1}q + \cdots$ we define the vector $\mathbf{k}_j = (k_{j,0},\dots,k_{j,n-1})^\top \in \mathbb{F}_q^n$.

The following criterion was first introduced in the context of applying digital nets to the numerical integration of smooth functions, see [2]: For $k \in \mathbb{N}$ and $\alpha \geq 1$ let $\mu_\alpha(k) = a_1 + \cdots + a_{\min(\nu,\alpha)}$, where $k = \kappa_1 q^{a_1-1} + \cdots + \kappa_\nu q^{a_\nu-1}$ with $0 < \kappa_1,\dots,\kappa_\nu < q$ and $1 \leq a_\nu < \cdots < a_1$. Further we set $\mu_\alpha(0) = 0$. For a vector $k = (k_1,\dots,k_s) \in \mathbb{N}_0^s$ we define $\mu_\alpha(k) = \mu_\alpha(k_1) + \cdots + \mu_\alpha(k_s)$. The following definition was motivated in [3].

**Definition 2.** Let $n,m,\alpha \in \mathbb{N}$, let $0 < \beta \leq \min(1, \alpha m/n)$ be a real number, and let $0 \leq t \leq \beta n$ be a non-negative integer. Let $\mathbb{F}_q$ be the finite field of prime power order $q$ and let $C_1,\dots,C_s \in \mathbb{F}_q^{n\times m}$ with $C_j = (\mathbf{c}_{j,1},\dots,\mathbf{c}_{j,n})^T$. If for all $1 \leq i_{j,\nu_j} < \cdots < i_{j,1}$, where $0 \leq \nu_j$ for all $j = 1,\dots,s$, with

$$\sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j,\alpha)} i_{j,l} \leq \beta n - t$$

the vectors

$$\mathbf{c}_{1,i_{1,\nu_1}},\dots,\mathbf{c}_{1,i_{1,1}},\dots,\mathbf{c}_{s,i_{s,\nu_s}},\dots,\mathbf{c}_{s,i_{s,1}}$$

are linearly independent over $\mathbb{F}_q$, then the digital net with generating matrices $C_1,\dots,C_s$ is called a digital $(t,\alpha,\beta,n\times m,s)$-net over $\mathbb{F}_q$.

If $t$ is the smallest nonnegative integer such that the digital net generated by $C_1,\dots,C_s$ is a digital $(t,\alpha,\beta,n\times m,s)$-net, then we call the digital net a strict digital $(t,\alpha,\beta,n\times m,s)$-net.

Note that the condition $\sum_{j=1}^{s}\sum_{l=1}^{\nu_j} i_{j,l} \leq \beta n - t$ implies that $i_{j,1} \leq n$, as $\beta \leq 1$ and $t \geq 0$.

Similarly, we can recall the definition of digital $(t,\alpha,\beta,\sigma,s)$-sequences over $\mathbb{F}_q$ from [2].

**Definition 3.** Let $\alpha,\sigma \geq 1$ and $t \geq 0$ be integers and let $0 < \beta \leq \alpha/\sigma$ be a real number. Let $\mathbb{F}_q$ be the finite field of prime power order $q$ and let $C_1,\dots,C_s \in \mathbb{F}_q^{\infty\times\infty}$ with $C_j = (\mathbf{c}_{j,1},\mathbf{c}_{j,2},\dots)^\top$. Further let $C_{j,\sigma m\times m}$ denote the left upper $\sigma m \times m$ submatrix of $C_j$. If for all $m > t/(\beta\sigma)$ the matrices $C_{1,\sigma m\times m},\dots,C_{s,\sigma m\times m}$ generate a digital $(t,\alpha,\beta,\sigma m\times m,s)$-net, then the digital sequence with generating matrices $C_1,\dots,C_s$ is called a digital $(t,\alpha,\beta,\sigma,s)$-sequence over $\mathbb{F}_q$.

If $t$ is the smallest nonnegative integer such that the digital sequence generated by $C_1,\ldots,C_s$ is a digital $(t,\alpha,\beta,\sigma,s)$-sequence, then we call the digital sequence a strict digital $(t,\alpha,\beta,\sigma,s)$-sequence.

*Remark 1.* In the following we explain the meaning of the parameters $t$, $\alpha$, $\beta$, $n$, $m$ and $s$ used in the context of generalized digital $(t,\alpha,\beta,n\times m,s)$-nets; see also [2, Remark 4.5]:

- $s$ denotes the dimensionality of the point set;
- the logarithm in base $q$ of the number of points is $m$, i.e., a digital $(t,\alpha,\beta,n\times m,s)$-net has $q^m$ points;
- $n$ denotes the number of rows of the generating matrices and therefore corresponds to the maximum number of non-zero digits in the base $q$ expansion of each coordinate of each point; hence $n$ determines how precise each point is placed in the unit cube, which has a direct influence on the convergence of the integration error as can be seen from the next point;
- $\beta n-t$ denotes the quality of the point set, which can be referred to as the strength of the net; in particular, the integration error is $\mathscr{O}(q^{-\beta n+t}(\beta n-t)^{\alpha s})$;
- digital $(t,\alpha,\beta,n\times m,s)$-nets were introduced in the context of numerical integration, where $\alpha$ is a variable parameter, which denotes the smoothness of the integrand. We assume that the smoothness $\alpha$ is not known explicitly.

Finally, following [2], we now recall a method of explicitly constructing digital $(t,\alpha,\beta,n\times m,s)$-nets, which was first presented in [2, Section 4.4]. This way we obtain digital $(t,\alpha,\min(1,\alpha/d),dm\times m,s)$-nets for all $\alpha\geq 1$, where $d\in\mathbb{N}$ is a parameter which can be chosen freely.

Let $d\geq 1$ and let $C_1,\ldots,C_{sd}$ be the generating matrices of a digital $(t',m,sd)$-net; we recall that many explicit examples of such generating matrices are known, see e.g., [6, 7, 8, 10, 12, 18] and the references therein. As we will see later, the choice of the underlying digital $(t',m,sd)$-net has a direct impact on the bound on the $t$-value of the digital $(t,\alpha,\min(1,\frac{\alpha}{d}),dm\times m,s)$-net, which was proven in [2]. Let $C_j=(\mathbf{c}_{j,1},\ldots,\mathbf{c}_{j,m})^\top$ for $j=1,\ldots,sd$; i.e., $\mathbf{c}_{j,l}$ are the row vectors of $C_j$. Now let the matrix $C_j^{(d)}$ consist of the first rows of the matrices $C_{(j-1)d+1},\ldots,C_{jd}$, then the second rows of $C_{(j-1)d+1},\ldots,C_{jd}$, and so on, in the order described in the following: The matrix $C_j^{(d)}$ is a $dm\times m$ matrix; i.e., $C_j^{(d)}=(\mathbf{c}_{j,1}^{(d)},\ldots,\mathbf{c}_{j,dm}^{(d)})^\top$, where $\mathbf{c}_{j,l}^{(d)}=\mathbf{c}_{u,v}$ with $l=(v-j)d+u$, $1\leq v\leq m$, and $(j-1)d<u\leq jd$ for $l=1,\ldots,dm$ and $j=1,\ldots,s$. We remark that this construction can be extended to digital $(t,\alpha,\beta,\sigma,s)$-sequences by letting $\widetilde{C}_j=(\widetilde{\mathbf{c}}_{j,1},\widetilde{\mathbf{c}}_{j,2},\ldots)^\top$, for $j=1,\ldots,sd$, denote the generating matrices of a digital $(t',sd)$-sequence; the resulting matrices $\widetilde{C}_j^{(d)}$, $j=1,\ldots,s$, are now $\infty\times\infty$ matrices, where again we have $\widetilde{C}_j^{(d)}=(\widetilde{\mathbf{c}}_{j,1}^{(d)},\widetilde{\mathbf{c}}_{j,2}^{(d)},\ldots)^\top$, where $\widetilde{\mathbf{c}}_{j,l}^{(d)}=\widetilde{\mathbf{c}}_{u,v}$ with $l=(v-j)d+u$, $v\geq 1$, and $(j-1)d<u\leq jd$ for $l=1,2,\ldots$ and $j=1,\ldots,s$.

The following result improves [2, Theorem 4.11] for some cases. For a proof see [4].

**Theorem 1.** *Let $d \geq 1$ be a natural number and let $C_1, \ldots, C_{sd}$ be the generating matrices of a digital $(t', m, sd)$-net over the finite field $\mathbb{F}_q$ of prime power order $q$. Let $C_1^{(d)}, \ldots, C_s^{(d)}$ be defined as above. Then for any $\alpha \in \mathbb{N}$, the matrices $C_1^{(d)}, \ldots, C_s^{(d)}$ are the generating matrices of a digital $(t, \alpha, \min(1, \alpha/d), dm \times m, s)$-net over $\mathbb{F}_q$ with*

$$t = \min(\alpha, d) \min\left(m, t' + \left\lfloor \frac{s(d-1)}{2} \right\rfloor\right). \tag{1}$$

*Furthermore, the matrices $\widetilde{C}_1^{(d)}, \ldots, \widetilde{C}_s^{(d)}$ obtained from the generating matrices $\widetilde{C}_1, \ldots, \widetilde{C}_{sd}$ of a digital $(t', sd)$-sequence over $\mathbb{F}_q$ are the generating matrices of a digital $(t, \alpha, \min(1, \alpha/d), d, s)$-sequence over $\mathbb{F}_q$ with*

$$t = \min(\alpha, d)\left(t' + \left\lfloor \frac{s(d-1)}{2} \right\rfloor\right).$$

In the following example we show that the above result cannot be improved on in general.

*Example 1.* Let $d = 2$ and $s = 1$ and generate a digital $(t, \alpha, \min(1, \alpha/2), 2m \times m, 1)$-net over $\mathbb{F}_q$ from a digital $(0, m, 2)$-net over $\mathbb{F}_q$ (such nets exist, for example one can take the Hammersley net). Then Theorem 1 implies that we can choose $t = \min(\alpha, 2)0 + \min(\alpha, 2)\lfloor 1 \cdot 1/2 \rfloor = 0$, which is already best possible.

On the other hand it can be checked that the bound on the $t$-value in Theorem 1 for particular digital nets is not necessarily best possible. That is, if we use a strict digital $(t', m, sd)$-net over $\mathbb{F}_q$ for the construction of the generating matrices $C_1^{(d)}, \ldots, C_s^{(d)}$, then these generating matrices do not necessarily generate a strict digital $(t, \alpha, \beta, n \times m, s)$-net over $\mathbb{F}_q$, where $t$ is given by (1). This is illustrated in the next example.

*Example 2.* The following matrices generate a strict digital $(1, 3, 4)$-net over $\mathbb{F}_2$ and stem from a Niederreiter-Xing sequence as implemented by Pirsic [16]:

$$C_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, C_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, C_3 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, C_4 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Using the method described in [2, Section 4.4] with $d = 2$, we construct the generating matrices $C_1^{(2)}$ and $C_2^{(2)}$, which are given by:

$$C_1^{(2)} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, C_2^{(2)} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

For any $\alpha \geq 2$, Theorem 1 yields a digital $(4,\alpha,1,6\times 3,2)$-net and for $\alpha = 1$ a digital $(2,1,1/2,6\times 3,2)$-net.

We now show that the exact $t$-value of this digital net is smaller than the one obtained from Theorem 1. It can be confirmed by inspection that the matrices $C_1^{(2)}$ and $C_2^{(2)}$ generate a digital $(2,\alpha,1,6\times 3,2)$-net for all $\alpha \geq 2$, by checking that for all $1 \leq i_{j,\nu_j} < \cdots < i_{j,1}$, where $0 \leq \nu_j$, $j = 1,2$, with

$$\sum_{j=1}^{2} \sum_{l=1}^{\min(\nu_j,\alpha)} i_{j,l} \leq 6-2 = 4$$

the vectors $\mathbf{c}_{1,i_{1,\nu_1}}^{(2)},\ldots,\mathbf{c}_{1,i_{1,1}}^{(2)},\mathbf{c}_{2,i_{2,\nu_2}}^{(2)},\ldots,\mathbf{c}_{2,i_{2,1}}^{(2)}$ are linearly independent over $\mathbb{F}_2$. Furthermore, it can be confirmed that the two matrices $C_1^{(2)}$ and $C_2^{(2)}$ do not generate a digital $(1,\alpha,1,6\times 3,2)$-net for any $\alpha \geq 2$, as for $\nu_1 = 0$, $\nu_2 = 2$, $i_{2,2} = 1$, $i_{2,1} = 4$, $\mathbf{c}_{2,i_{2,1}}^{(2)}$ and $\mathbf{c}_{2,i_{2,2}}^{(2)}$ are linearly dependent. Hence, for any $\alpha \geq 2$, the matrices $C_1^{(2)}$ and $C_2^{(2)}$ generate a strict digital $(2,\alpha,1,6\times 3,2)$-net.

For $\alpha = 1$ on the other hand, it can be checked that the matrices $C_1^{(2)}$ and $C_2^{(2)}$ generate a strict digital $(0,1,1/2,6\times 3,2)$-net.

Thus, for this example, Theorem 1 does not yield the best possible result for any $\alpha \geq 1$.

Next we present an example which might be counterintuitive at first: We present a strict digital $(2,3,4)$-net, which generates a strict digital $(1,\alpha,1,6\times 3,2)$-net for any $\alpha \geq 2$, and, for $\alpha = 1$, a strict digital $(0,1,1/2,6\times 3,2)$-net.

*Example 3.* The following matrices generate a strict digital $(2,3,4)$-net over $\mathbb{F}_2$:

$$K_1 = \begin{pmatrix} 1&1&0\\ 1&0&0\\ 1&1&0 \end{pmatrix}, K_2 = \begin{pmatrix} 1&0&1\\ 1&0&1\\ 0&0&0 \end{pmatrix}, K_3 = \begin{pmatrix} 0&0&1\\ 1&0&0\\ 0&0&1 \end{pmatrix}, K_4 = \begin{pmatrix} 0&1&0\\ 0&1&0\\ 0&0&0 \end{pmatrix}.$$

Using the method described in [2, Section 4.4] with $d = 2$, we construct the generating matrices $K_1^{(2)}$ and $K_2^{(2)}$, which are given by:

$$K_1^{(2)} = \begin{pmatrix} 1&1&0\\ 1&0&1\\ 1&0&0\\ 1&0&1\\ 1&1&0\\ 0&0&0 \end{pmatrix}, K_2^{(2)} = \begin{pmatrix} 0&0&1\\ 0&1&0\\ 1&0&0\\ 0&1&0\\ 0&0&1\\ 0&0&0 \end{pmatrix}.$$

For any $\alpha \geq 2$, Theorem 1 yields a digital $(6,\alpha,1,6\times 3,2)$-net, and for $\alpha = 1$ a digital $(3,1,1/2,6\times 3,2)$-net.

As in Example 2, it can be confirmed by inspection that the matrices $K_1^{(2)}$ and $K_2^{(2)}$ generate a digital $(1,\alpha,1,6\times 3,2)$-net for all $\alpha \geq 2$. Furthermore, it can be confirmed that the two matrices $K_1^{(2)}$ and $K_2^{(2)}$ do not generate a digital $(0,\alpha,1,6\times 3,2)$-net for $\alpha \geq 2$, as for $\nu_1 = 2$, $\nu_2 = 2$, $i_{1,2} = 1$, $i_{1,1} = 2$, $i_{2,2} = 1$ and $i_{2,1} = 2$, $\mathbf{k}_{1,i_{1,2}}^{(2)}$, $\mathbf{k}_{1,i_{1,1}}^{(2)}$, $\mathbf{k}_{2,i_{2,2}}^{(2)}$ and $\mathbf{k}_{2,i_{2,1}}^{(2)}$ are linearly dependent, where $\mathbf{k}_{j,i}^{(2)}$ denotes the $i$th row of the matrix $K_j^{(2)}$.

For $\alpha = 1$ on the other hand, it can be checked that the matrices $K_1^{(2)}$ and $K_2^{(2)}$ generate a strict digital $(0,1,1/2,6\times 3,2)$-net.

The last two examples show that Theorem 1 does not always yield the best possible bounds on the $t$-value for digital $(t,\alpha,\beta,n\times m,s)$-nets constructed from particular classical digital nets. (This could mean that it might be possible to improve the bound on the $t$-value for generalized digital nets constructed from particular classical nets (or sequences).) On the other hand, at least for digital $(t,\alpha,\beta,\sigma,s)$-sequences, we will see below that Theorem 1 does yield the asymptotically optimal dependence of the $t$-value on $\alpha$ and $s$, see Theorem 7 below.

*Remark 2.* Note that even though the strict digital $(1,3,4)$-net used in Example 2 has a better $t$-value (in the classical sense) than the strict digital $(2,3,4)$-net in Example 3, the latter generates the better digital $(t,\alpha,1,6\times 3,2)$-net for any $\alpha \geq 2$, as measured by the generalized $t$-value. However, it is possible to find a strict digital $(1,3,4)$-net which generates a strict digital $(1,\alpha,1,6\times 3,2)$-net for any $\alpha \geq 2$. Consider for example

$$\tilde{K}_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \tilde{K}_2 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \tilde{K}_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tilde{K}_4 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

*Remark 3.* It can be checked that the matrices $C_1^{(2)}$ and $C_2^{(2)}$ from Example 2 can also be interpreted as generating matrices of a digital $(0,3,2)$-net over $\mathbb{F}_2$. However, if we set $\tilde{C}_2 = C_2^{(2)}$, but

$$\tilde{C}_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

we have an example of a strict digital $(2,\alpha,1,6\times 3,2)$-net over $\mathbb{F}_2$, $\alpha \geq 2$, which is a strict digital $(1,3,2)$-net.

## 3 Equidistribution properties of generalized nets and sequences

Generalized digital nets and sequences, as introduced in [2], rely on linear algebra over finite fields. The quality of such point sets is determined by linear in-dependence properties of the generating matrices. In this section we remove this restriction by introducing the essential geometrical properties satisfied by digital $(t,\alpha,\beta,n\times m,s)$-nets and digital $(t,\alpha,\beta,\sigma,s)$-sequences. This is analogous to the link between $(t,m,s)$-nets and digital $(t,m,s)$-nets in the classical theory (or $(t,s)$-sequences and digital $(t,s)$-sequences), where the former includes the latter as a special case and $(t,m,s)$-nets (and $(t,s)$-sequences) are defined using only geometrical features of the point set.

## *3.1 Definition of* $(t,\alpha,\beta,n,m,s)$*-nets and* $(t,\alpha,\beta,\sigma,s)$*-sequences*

We recall that the definition of $(t,m,s)$-nets is based on the concept of an elementary interval, see e.g. [10]. In the following we introduce a concept analogous to that of an elementary interval, namely that of a generalized elementary interval. Before we do so we need some notation: let $\nu = (\nu_1,\ldots,\nu_s)$, let $|\nu|_1 = \sum_{j=1}^{s} \nu_j$, let $\mathbf{i}_\nu = (i_{1,1},\ldots,i_{1,\nu_1},\ldots,i_{s,1},\ldots,i_{s,\nu_s})$, let $\mathbf{a}_\nu \in \{0,\ldots,q-1\}^{|\nu|_1}$, and let $\mathbf{a}_\nu = (a_{1,i_{1,1}},\ldots,a_{1,i_{1,\nu_1}},\ldots,a_{s,i_{s,1}},\ldots,a_{s,i_{s,\nu_s}})$, where the components $i_{j,l}$ and $a_{j,l}$, $l=1,\ldots,\nu_j$, do not appear in the vectors $\mathbf{i}_\nu$ and $\mathbf{a}_\nu$ in case $\nu_j = 0$.

By a *generalized elementary interval* we mean a subset of $[0,1)^s$ of the form

$$J(\mathbf{i}_\nu,\mathbf{a}_\nu) = \prod_{j=1}^{s} \bigcup_{\substack{a_{j,l}=0 \\ l\in\{1,\ldots,n\}\setminus\left\{i_{j,1},\ldots,i_{j,\nu_j}\right\}}}^{q-1} \left[\frac{a_{j,1}}{q}+\cdots+\frac{a_{j,n}}{q^n}, \frac{a_{j,1}}{q}+\cdots+\frac{a_{j,n}}{q^n}+\frac{1}{q^n}\right),$$

where $q\geq 2$ is an integer and where for $j=1,\ldots,s$ we have $1\leq i_{j,\nu_j} < \cdots < i_{j,1} \leq n$ in case $\nu_j > 0$ and $\{i_{j,1},\ldots,i_{j,\nu_j}\} = \emptyset$ in case $\nu_j = 0$.

We note that a generalized elementary interval is not always an elementary interval, but can be a union of several elementary intervals, see for example Figure 1.

Generalized elementary intervals posses properties similar to those of classical elementary intervals as we show in the following.

**Lemma 1.** *Let* $\nu \in \{0,\ldots,n\}^s$ *and* $\mathbf{i}_\nu$ *be defined as above and fixed. Then the generalized elementary intervals* $J(\mathbf{i}_\nu,\mathbf{a}_\nu)$ *for* $\mathbf{a}_\nu \in \{0,\ldots,q-1\}^{|\nu|_1}$*, form a partition of* $[0,1)^s$*, i.e.* $\bigcup_{\mathbf{a}_\nu\in\{0,\ldots,q-1\}^{|\nu|_1}} J(\mathbf{i}_\nu,\mathbf{a}_\nu) = [0,1)^s$ *and* $J(\mathbf{i}_\nu,\mathbf{a}_\nu)\cap J(\mathbf{i}_\nu,\mathbf{a}'_\nu) = \emptyset$*, for all* $\mathbf{a}_\nu \neq \mathbf{a}'_\nu \in \{0,\ldots,q-1\}^{|\nu|_1}$.

*Proof.* First we have

$$
\begin{aligned}
&\bigcup_{\mathbf{a}_\nu \in \{0,\dots,q-1\}^{|\nu|_1}} J(\mathbf{i}_\nu, \mathbf{a}_\nu) \\
&= \prod_{j=1}^{s} \bigcup_{\substack{a_{j,l}=0 \\ l\in\{1,\dots,n\}}}^{q-1} \left[\frac{a_{j,1}}{q} + \cdots + \frac{a_{j,n}}{q^n}, \frac{a_{j,1}}{q} + \cdots + \frac{a_{j,n}}{q^n} + \frac{1}{q^n}\right) \\
&= [0,1)^s.
\end{aligned}
$$

To show the second part, we note that, for $\mathbf{i}_\nu$ fixed and $\mathbf{a}_\nu \neq \mathbf{a}'_\nu$, there exists a $j \in \{1,\dots,s\}$, and a $k \in \left\{i_{j,1},\dots,i_{j,\nu_j}\right\}$, such that $a_{j,k} \neq a'_{j,k}$. Let $\mathbf{x} = (x_1,\dots,x_s)$ where each coordinate $x_j$, $j = 1,\dots,s$, has base $q$ expansion $x_j = x_{j,1}q^{-1} + x_{j,2}q^{-2} + \dots$ (we assume that for each $j \in \{1,\dots,s\}$ infinitely many $x_{j,k} \neq q-1$). Then $\mathbf{x} \in J(\mathbf{i}_\nu, \mathbf{a}_\nu)$ if and only if for all $j = 1,\dots,s$ and all $k \in \{i_{j,1},\dots,i_{j,\nu_j}\}$ we have $x_{j,k} = a_{j,k}$. But as there exists a $j$ and $k$ such that $a_{j,k} \neq a'_{j,k}$, $\mathbf{x}$ cannot be in $J(\mathbf{i}_\nu, \mathbf{a}_\nu)$ and $J(\mathbf{i}_\nu, \mathbf{a}'_\nu)$ simultaneously. Hence $J(\mathbf{i}_\nu, \mathbf{a}_\nu) \cap J(\mathbf{i}_\nu, \mathbf{a}'_\nu) = \emptyset$ and the result follows.

$\square$

In the following lemma, we compute the volume of a generalized elementary interval.

**Lemma 2.** *Let $\nu$, $\mathbf{i}_\nu$ and $\mathbf{a}_\nu$ be as above. Then the volume of $J(\mathbf{i}_\nu, \mathbf{a}_\nu)$ is $q^{-|\nu|_1}$.*

*Proof.* Let $\nu$ and $\mathbf{i}_\nu$ be fixed. Then we have seen in Lemma 1 that the $J(\mathbf{i}_\nu, \mathbf{a}_\nu)$, $\mathbf{a}_\nu \in \{0,\dots,q-1\}^{|\nu|_1}$ form a partition of $[0,1)^s$. From the definition of generalized elementary intervals one can see that $\mathrm{Vol}(J(\mathbf{i}_\nu, \mathbf{a}_\nu)) = \mathrm{Vol}(J(\mathbf{i}_\nu, \mathbf{a}'_\nu))$ for all $\mathbf{a}_\nu, \mathbf{a}'_\nu \in \{0,\dots,q-1\}^{|\nu|_1}$, where $\mathrm{Vol}(J)$ denotes the volume of an interval $J$, as the intervals $J(\mathbf{i}_\nu, \mathbf{a}_\nu)$ and $J(\mathbf{i}_\nu, \mathbf{a}'_\nu)$ are only shifted versions of each other. Hence

$$
\mathrm{Vol}(J(\mathbf{i}_\nu, \mathbf{a}_\nu)) = \frac{1}{|\{\mathbf{a}_\nu \in \{0,\dots,q-1\}^{|\nu|_1}\}|} = \frac{1}{q^{|\nu|_1}}.
$$

$\square$

We are now in a position to define a $(t,\alpha,\beta,n,m,s)$-net, which is based on the concept of a generalized elementary interval and Lemma 2.

**Definition 4.** Let $n,m,\alpha \geq 1$ be natural numbers, let $0 < \beta \leq 1$ be a real number, and let $0 \leq t \leq \beta n$ be an integer. Let $q \geq 2$ be an integer and $P = \left\{x_0,\dots,x_{q^m-1}\right\} \subseteq [0,1)^s$ be a point set in the $s$-dimensional unit cube, $s \geq 1$. We say that $P$ is a $(t,\alpha,\beta,n,m,s)$-net (in base $q$), if for all integers $1 \leq i_{j,\nu_j} < \cdots < i_{j,1}$, where $\nu_j \geq 0$, with

$$
\sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j,\alpha)} i_{j,l} \leq \beta n - t,
$$

where for $\nu_j = 0$ we set the empty sum $\sum_{l=1}^{0} i_{j,l} = 0$, the generalized elementary interval $J(\mathbf{i}_\nu, \mathbf{a}_\nu)$ contains exactly $q^{m-|\nu|_1}$ points of $P$ for each $\mathbf{a}_\nu \in \{0,\dots,q-1\}^{|\nu|_1}$.

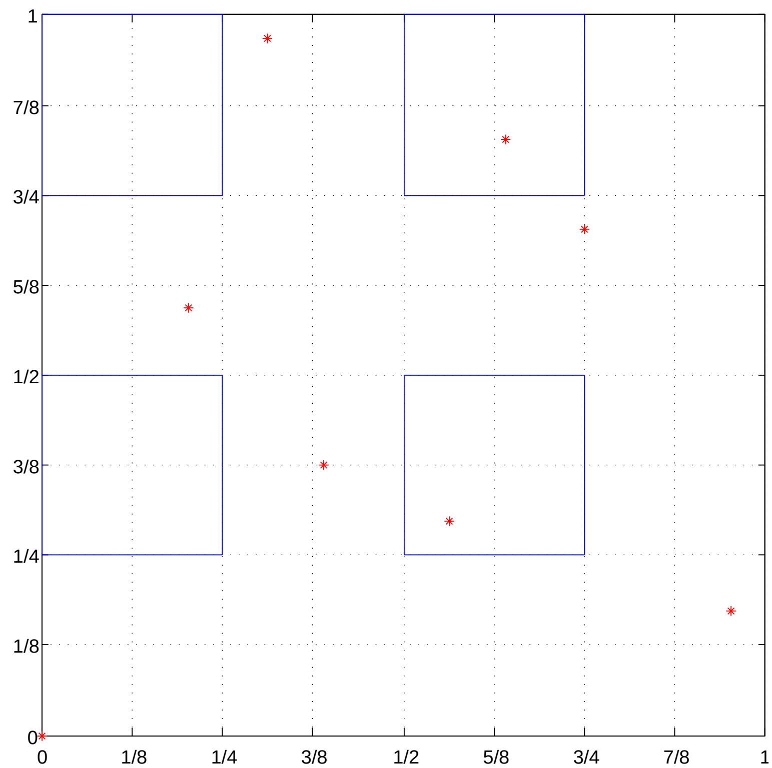


**Fig. 1** The picture shows a $(2,\alpha,1,6,3,2)$-net in base 2 for any $\alpha \geq 2$ and a generalized elementary interval $J(\mathbf{i}_\nu,\mathbf{a}_\nu)$, where $\nu_1 = \nu_2 = 1$, $i_{1,1} = i_{2,1} = 2$, and $a_{i_{1,1}} = 0$ and $a_{i_{2,1}} = 1$.

*Remark 4.* Note that $q^{m-|\nu|_1} = q^m \mathrm{Vol}(J(\mathbf{i}_\nu,\mathbf{a}_\nu))$. For an interval $J \subseteq [0,1)^s$ and a point set $P \subset [0,1)^s$, let $|P(J)|$ denote the number of points of $P$ in $J$. Then Definition 4 says that the proportion of points of $P$ in $J(\mathbf{i}_\nu,\mathbf{a}_\nu)$, which is given by $|P(J(\mathbf{i}_\nu,\mathbf{a}_\nu))|/|P([0,1)^s)|$, equals the volume of $J(\mathbf{i}_\nu,\mathbf{a}_\nu)$.

*Remark 5.* Note that $(t,\alpha,\beta,n,m,s)$-nets can only exist for parameters $t$, $\alpha$, $\beta$, $n$, $m$, $s$ where the definition implies that $\nu_1 + \cdots + \nu_s \leq m$.

Consider for example the choice of parameters $\beta = 1$, $t = \alpha = s = 2$, $m = 3$ and $n = 6$; such a $(2,2,1,6,3,2)$-net can exist, since if $\nu_1 + \nu_2 > 3$ we have for all choices of $1 \leq i_{j,\nu_j} < \cdots < i_{j,1} \leq 6$, for $j = 1,2$, that $\sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j,\alpha)} i_{j,l} > 4 = \beta n - t$. (On the other hand, that does not imply that such a net really does exist, it only allows for the possibility to exist.)

But a $(0,2,1,6,3,2)$-net, i.e. we set $t = 0$ and leave the remaining parameters unchanged, cannot exist, since we could choose $\nu_1 = \nu_2 = 2$, $i_{1,1} = i_{2,1} = 2$ and $i_{1,2} = i_{2,2} = 1$, in which case we have $i_{1,1} + i_{1,2} + i_{2,1} + i_{2,2} = 6 = \beta n - t$, and thereby obtain a generalized elementary interval which has to contain exactly $q^{m-\nu_1-\nu_2} = q^{-1}$ points, which is of course absurd. Hence $t = 0$ is not possible for this choice of parameters. (Regarding $t = 1$, we have explicitly constructed digital $(1,2,1,6\times 3,2)$-nets in Example 3 and Remark 2, which by Theorem 6 below also form a $(1,2,1,6,3,2)$-net.)

*Remark 6.* We obtain the definition of a classical $(t,m,s)$-net from Definition 4 by setting $\alpha = \beta = 1$, $n = m$, and considering all $\nu_1,\ldots,\nu_s \ge 0$ so that $\sum_{j=1}^{s}\nu_j \le m-t$, where we set $i_{j,k} = \nu_j - k + 1$ for $k = 1,\ldots,\nu_j$. Hence a $(t,1,1,m,m,s)$-net is a $(t,m,s)$-net.

We shall now discuss the additional parameters $\alpha,\beta$, and $n$, which do not appear in the definition of classical $(t,m,s)$-nets. The case $\alpha = 1$ is strongly related to classical $(t,m,s)$-nets. We can, w.l.o.g., choose $\nu_j$, $j = 1,\ldots,s$ so that $\sum_{j=1}^{s}\nu_j = \lfloor\beta n\rfloor - t$ and set $i_{j,l} = \nu_j + 1 - l$ for $l = 1,\ldots,\nu_j$, as in this case we obtain the most stringent condition on the points, i.e., all other conditions are automatically included in this choice of the $i_{j,l}$. Then a $(t,1,\beta,n,m,s)$-net is a classical $(t',m,s)$-net with $t' = m - \lfloor\beta n\rfloor + t$.

We have the following theorem.

**Theorem 2.** *Assume that $n,m,\alpha \in \mathbb{N}$, $0 < \beta \le 1$ a real number, and $0 \le t \le \beta n$ an integer, such that there exists a $(t,\alpha,\beta,n,m,s)$-net in base $q$. For $1 \le j_0 \le s$ let $0 \le \ell_{j_0} < j_0$ be given by $\ell_{j_0} \equiv m \pmod{j_0}$. Then for $j_0 = 1,\ldots,s$ we have*

$$\beta n - t < \alpha m - j_0\frac{\alpha(\alpha-1)}{2} + \alpha, \qquad \text{for } m \ge \alpha j_0,$$

*and*

$$\beta n - t < \frac{1}{2}\alpha j_0\left\lfloor\frac{m}{j_0}\right\rfloor + (\ell_{j_0}+1)\left(\left\lfloor\frac{m}{j_0}\right\rfloor + 1\right), \qquad \text{for } m < \alpha j_0.$$

*Proof.* As elaborated in Remark 5, for every choice of $1 \le i_{j,\nu_j} < \cdots < i_{j,1}$, $\nu_j \ge 0$, for $j = 1,\ldots,s$, with $\sum_{j=1}^{s}\sum_{l=1}^{\min(\nu_j,\alpha)} i_{j,l} \le \beta n - t$, we must have that $|\nu|_1 \le m$.

Let $1 \le j_0 \le s$ and let

$$\nu_j = \begin{cases} \lfloor m/j_0\rfloor + 1 & \text{for } 1 \le j \le \ell_{j_0}+1, \\ \lfloor m/j_0\rfloor & \text{for } \ell_{j_0}+2 \le j \le j_0, \\ 0 & \text{for } j_0+1 \le j \le s. \end{cases}$$

Further set $i_{j,l} = \nu_j + 1 - l$ for $l = 1,\ldots,\nu_j$ for $j = 1,\ldots,j_0$. Note that for this choice of $\nu_1,\ldots,\nu_s$ we have

$$|\nu|_1 = j_0\left\lfloor\frac{m}{j_0}\right\rfloor + \ell_{j_0} + 1 = j_0\frac{m-\ell_{j_0}}{j_0} + \ell_{j_0} + 1 = m+1.$$

Consider the case where $\alpha \le \lfloor m/j_0\rfloor$. Then

$$\begin{aligned}\sum_{j=1}^{s}\sum_{l=1}^{\min(\nu_j,\alpha)} i_{j,l} &= j_0\left(\left\lfloor\frac{m}{j_0}\right\rfloor+\left\lfloor\frac{m}{j_0}\right\rfloor-1+\cdots+\left\lfloor\frac{m}{j_0}\right\rfloor-(\alpha-1)\right)+\alpha(\ell_{j_0}+1)\\ &= \alpha j_0\left\lfloor\frac{m}{j_0}\right\rfloor - j_0\frac{\alpha(\alpha-1)}{2}+\alpha(\ell_{j_0}+1)\\ &= \alpha j_0\frac{m-\ell_{j_0}}{j_0}-j_0\frac{\alpha(\alpha-1)}{2}+\alpha\ell_{j_0}+\alpha\\ &= \alpha m - j_0\frac{\alpha(\alpha-1)}{2}+\alpha.\end{aligned}$$

Thus we get a contradiction if the last term is smaller or equal to $\beta n - t$ and hence the first result follows.

Now we consider the case where $\alpha \ge \lfloor m/j_0\rfloor + 1$. Then

$$\begin{aligned}\sum_{j=1}^{s}\sum_{l=1}^{\min(\nu_j,\alpha)} i_{j,l} &= j_0\left(\left\lfloor\frac{m}{j_0}\right\rfloor+\left\lfloor\frac{m}{j_0}\right\rfloor-1+\cdots+1\right)+(\ell_{j_0}+1)\left(\left\lfloor\frac{m}{j_0}\right\rfloor+1\right)\\ &= j_0\frac{\lfloor m/j_0\rfloor(\lfloor m/j_0\rfloor+1)}{2}+(\ell_{j_0}+1)\left(\left\lfloor\frac{m}{j_0}\right\rfloor+1\right)\\ &\le \frac{1}{2}\alpha j_0\left\lfloor\frac{m}{j_0}\right\rfloor+(\ell_{j_0}+1)\left(\left\lfloor\frac{m}{j_0}\right\rfloor+1\right).\end{aligned}$$

Again we get a contradiction if the last term is smaller or equal to $\beta n - t$ and hence also the second result follows.

□

Note, Theorem 2 implies for $\alpha = 1,2$ that $\beta n - t < \alpha m + 1$ (choose $j_0 = 1$) and, based on the proof of Theorem 2, one can show that $\beta n - t < \alpha m$ for $\alpha \ge 3$ (choose $j_0 = 1$). Thus, as $\beta n - t < \alpha m + 1$, we can w.l.o.g. choose $\beta$ and $n$ such that $\beta n < \alpha m + 1$ (for $\beta n \ge \alpha m + 1$ we must have $t > 0$, hence we do not exclude any cases by choosing $\beta n < \alpha m + 1$), or if $\beta$ is such that $\beta n$ is an integer, we have $\beta \le \alpha m/n$.

Choosing $j_0 = s$ in Theorem 2 and estimating $\ell_{j_0} + 1 \le j_0$, we obtain the following corollary.

**Corollary 1.** *Assume that $n, m, \alpha \in \mathbb{N}$, $0 < \beta \le 1$ a real number, and $0 \le t \le \beta n$ an integer, such that there exists a $(t,\alpha,\beta,n,m,s)$-net in base $q$. Then we have*

$$\beta n - t < \alpha m - s\frac{\alpha(\alpha-1)}{2}+\alpha, \qquad \text{for } m \ge \alpha s,$$

*and*

$$\beta n - t < \frac{1}{2}\alpha m + m + s, \qquad \text{for } m < \alpha s.$$

As in the classical case, we can also define sequences.

**Definition 5.** Let $\alpha, \sigma \geq 1$, $t \geq 0$ be integers, and $0 < \beta \leq 1$ be a real number. Let $S = \{x_0, x_1, \dots\}$ be a sequence of points in $[0,1)^s$. Then $S$ is a $(t, \alpha, \beta, \sigma, s)$-sequence in base $q$ if for all $k \geq 0$ and $m > t/(\beta\sigma)$ we have that $x_{kq^m}, x_{kq^m+1}, \dots, x_{(k+1)q^m-1}$ is a $(t, \alpha, \beta, \sigma m, m, s)$-net in base $q$.

*Remark 7.* We obtain the definition of a classical $(t,s)$-sequence from Definition 5 and Remark 6 by setting $\alpha = \beta = \sigma = 1$. Hence a $(t,1,1,1,s)$-sequence is a $(t,s)$-sequence.

### 3.2 Some properties of $(t, \alpha, \beta, n, m, s)$-nets and $(t, \alpha, \beta, \sigma, s)$-sequences

In this subsection we establish a few propagation rules for $(t, \alpha, \beta, n, m, s)$-nets and $(t, \alpha, \beta, \sigma, s)$-sequences in base $q$. Furthermore, we establish that every digital $(t, \alpha, \beta, n \times m, s)$-net over $\mathbb{F}_q$ is a $(t, \alpha, \beta, n, m, s)$-net in base $q$ and that every digital $(t, \alpha, \beta, \sigma, s)$-sequence over $\mathbb{F}_q$ is also a $(t, \alpha, \beta, \sigma, s)$-sequence in base $q$. Finally, we produce lower and upper bounds on the quality parameter $t$ for $(t, \alpha, \beta, \sigma, s)$-sequences.

The following theorem is in analogy to [2, Theorem 4.10].

**Theorem 3.** *Let $P$ be a $(t, \alpha, \beta, n, m, s)$-net in base $q$ and let $S$ be a $(t, \alpha, \beta, \sigma, s)$-sequence in base $q$. Then we have the following:*

(i) *$P$ is a $(t', \alpha, \beta', n, m, s)$-net for all $0 < \beta' \leq \beta$ and all $t \leq t' \leq \beta' n$, and $S$ is a $(t', \alpha, \beta', \sigma, s)$-sequence for all $0 < \beta' \leq \beta$ and all $t \leq t'$.*
(ii) *$P$ is a $(t', \alpha', \beta', n, m, s)$-net for all $\alpha' \geq 1$ where $\beta' = \beta \min(\alpha, \alpha')/\alpha$ and $t' = \lceil t \min(\alpha, \alpha')/\alpha \rceil$, and $S$ is a $(t', \alpha', \beta', \sigma, s)$-sequence for all $\alpha' \geq 1$ where $\beta' = \beta \min(\alpha, \alpha')/\alpha$ and where $t' = \lceil t \min(\alpha, \alpha')/\alpha \rceil$.*
(iii) *Any $(t, \alpha, \beta, \sigma, s)$-sequence is a $(t, \alpha, \beta, \sigma', s)$-sequence for all $1 \leq \sigma' \leq \sigma$.*
(iv) *Any $(t, \alpha, \beta, n, m, s)$-net is a classical $(m - \lfloor \beta n/\alpha \rfloor + \lceil t/\alpha \rceil, m, s)$-net and any $(t, \alpha, \beta, \sigma, s)$-sequence with $\alpha = \beta\sigma$ is a classical $(\lceil t/\alpha \rceil, s)$-sequence.*

*Proof.* For the first part note that $\beta' n - t' \leq \beta n - t$ and hence the condition on $P$ in Definition 4 is either the same or weaker. The same holds for $S$, hence the first part follows.

To prove the second part we consider firstly the case $\alpha' \geq \alpha$. Let $1 \leq i_{j,\nu_j} < \dots < i_{j,1}$, $\nu_j \geq 0$, for $j = 1, \dots, s$ with

$$\sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j, \alpha')} i_{j,l} \leq \beta n - t.$$

As

$$\sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j, \alpha)} i_{j,l} \leq \sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j, \alpha')} i_{j,l}$$

and $P$ is a $(t,\alpha,\beta,n,m,s)$-net, it follows that $J(\mathbf{i}_\nu,\mathbf{a}_\nu)$ contains $q^{m-|\nu|_1}$ points for all admissible $\mathbf{a}_\nu$ and hence this case follows for nets.

Let now $\alpha' < \alpha$ and assume

$$\sum_{j=1}^{s}\sum_{l=1}^{\min(\nu_j,\alpha')} i_{j,l} \le \beta' n - t' = \frac{\alpha'}{\alpha}\beta n - \left\lceil t\frac{\alpha'}{\alpha}\right\rceil.$$

As

$$\frac{1}{\alpha}\sum_{j=1}^{s}\sum_{l=1}^{\min(\nu_j,\alpha)} i_{j,l} \le \frac{1}{\alpha'}\sum_{j=1}^{s}\sum_{l=1}^{\min(\nu_j,\alpha')} i_{j,l},$$

it follows that

$$\sum_{j=1}^{s}\sum_{l=1}^{\min(\nu_j,\alpha)} i_{j,l} \le \frac{\alpha}{\alpha'}(\beta' n - t') \le \beta n - t.$$

As $P$ is a $(t,\alpha,\beta,n,m,s)$-net, it follows that $J(\mathbf{i}_\nu,\mathbf{a}_\nu)$ contains exactly $q^{m-|\nu|_1}$ points for all admissible $\mathbf{a}_\nu$, completing the proof for nets. For sequences the result follows from the result for nets and Definition 5.

For the third part we have to show that every point set $x_{kq^m},\ldots,x_{(k+1)q^m-1}$ is a $(t,\alpha,\beta,\sigma' m,m,s)$-net. We know that this point set is a $(t,\alpha,\beta,\sigma m,m,s)$-net from Definition 5. As $\sigma' m - t \le \sigma m - t$ this follows as the condition on the points $x_{kq^m},\ldots,x_{(k+1)q^m-1}$ can only become weaker, which implies the result.

For the last part we use (ii), which shows that every $(t,\alpha,\beta,n,m,s)$-net $P$ is also a $(\lceil t/\alpha\rceil,1,\beta/\alpha,n,m,s)$-net. After Remark 6, it was shown that Definition 4 implies that a $(t,1,\beta,n,m,s)$-net is a $(t',m,s)$-net, with $t' = m - \lfloor \beta n\rfloor + t$, hence $P$ is also a classical $(t',m,s)$-net, where

$$t' = m - \left\lfloor \frac{\beta}{\alpha} n\right\rfloor + \left\lceil \frac{t}{\alpha}\right\rceil.$$

Now consider a $(t,\alpha,\beta,\sigma,s)$-sequence $x_0,x_1,\ldots$. For any $k\ge 0$, the set of points $x_{kq^m},\ldots,x_{(k+1)q^m-1}$ forms a $(t,\alpha,\beta,\sigma m,m,s)$-net. Hence the above result implies that this is a classical $(t',m,s)$-net where

$$t' = m - \left\lfloor \frac{\beta}{\alpha}\sigma m\right\rfloor + \left\lceil \frac{t}{\alpha}\right\rceil = \left\lceil \frac{t}{\alpha}\right\rceil.$$

As $x_{kq^m},\ldots,x_{(k+1)q^m-1}$ is a classical $(t',m,s)$-net for all $k\ge 0$, the result follows.

$\square$

*Remark 8.* By Theorem 3, a $(2,\alpha,1,6,3,2)$-net, $\alpha\ge 2$, is a classical $(4-\lfloor\frac{6}{\alpha}\rfloor,3,2)$-net. By the forthcoming Theorem 6, the digital $(2,\alpha,1,6\times 3,2)$-net from Remark 3 is a $(2,\alpha,1,6,3,2)$-net, hence we have an example of a $(2,\alpha,1,6,3,2)$-net which is a strict $(1,3,2)$-net. See also Figure 1 for an example of a $(2,\alpha,1,6,3,2)$-net, which is a $(0,3,2)$-net.

In part (iv) of the above theorem we had the restriction that $\alpha = \beta\sigma$. If $S$ is a $(t,\alpha,\beta,\sigma,s)$-sequence with $\alpha > \beta\sigma$, then we cannot use (iv) of the above theorem to imply that $S$ is a classical $(t',s)$-sequence, as then we would obtain a $t'$-value of the subnets $x_{kq^m},\ldots,x_{(k+1)q^m-1}$ which grows with $m$. Hence we do not obtain a classical sequence this way. On the other hand, we always have $\alpha \geq \beta\sigma$, as we show in the following theorem.

**Theorem 4.** *Assume that $t,\alpha,\sigma,s \in \mathbb{N}$, and $\beta \in \mathbb{R}$, $0 < \beta \leq 1$ are such that there exists a $(t,\alpha,\beta,\sigma,s)$-sequence. Then $\beta\sigma \leq \alpha$.*

*Proof.* Let $x_0,x_1,\ldots$ be a $(t,\alpha,\beta,\sigma,s)$-sequence. Then the set of points $x_0,\ldots,x_{q^m-1}$ forms a $(t,\alpha,\beta,\sigma m,m,s)$-net for all $m > t/(\beta\sigma)$.

Assume to the contrary that $\alpha < \beta\sigma$. As $\beta\sigma m - t < \alpha m + 1$, which was shown after the proof of Theorem 2, we can choose an $m$ large enough to obtain a contradiction. Hence $\beta\sigma \leq \alpha$.

□

Digital sequences for which $\alpha = \beta\sigma$ are of interest, as in this case we get the optimal rate of convergence of the integration error for functions with square integrable partial mixed derivatives of order $\alpha$ in each variable, whereas for $\alpha > \beta\sigma$ we do not get the optimal rate, see [2]. But for the case $\alpha = \beta\sigma$ we get the following bound on the value of $t$ from Theorem 2.

**Theorem 5.** *Assume that $t,\alpha,\sigma,s \in \mathbb{N}$, and $\beta \in \mathbb{R}$, $0 < \beta \leq 1$, are such that $\alpha = \beta\sigma$ and such that there exists a $(t,\alpha,\beta,\sigma,s)$-sequence. Then for all $\alpha \geq 2$ we have*

$$t > s\frac{\alpha(\alpha-1)}{2} - \alpha.$$

*Proof.* Let $m_0 = \alpha s$. Then the first $q^{m_0}$ points of a $(t,\alpha,\beta,\sigma,s)$-sequence form a $(t,\alpha,\beta,\sigma m_0,m_0,s)$-net. By Corollary 1 we obtain that

$$\beta\sigma m_0 - t < \alpha m_0 - s\frac{\alpha(\alpha-1)}{2} + \alpha.$$

By substituting $\alpha$ for $\beta\sigma$ in the last equation we obtain the result.

□

The next theorem establishes that a digital $(t,\alpha,\beta,n\times m,s)$-net over $\mathbb{F}_q$ is a $(t,\alpha,\beta,n,m,s)$-net in base $q$ and analogously for sequences. This also yields explicit constructions of $(t,\alpha,\beta,n,m,s)$-nets and $(t,\alpha,\beta,\sigma,s)$-sequences as digital constructions are known from [2].

**Theorem 6.** *Every digital $(t,\alpha,\beta,n\times m,s)$-net over $\mathbb{F}_q$ is a $(t,\alpha,\beta,n,m,s)$-net in base $q$ and every digital $(t,\alpha,\beta,\sigma,s)$-sequence over $\mathbb{F}_q$ is a $(t,\alpha,\beta,\sigma,s)$-sequence in base $q$.*

*Proof.* Assume we are given an arbitrary generalized elementary interval

$$J(\mathbf{i}_\nu,\mathbf{a}_\nu) = \prod_{j=1}^{s} \bigcup_{\substack{a_{j,l}=0 \\ l\in\{1,\ldots,n\}\setminus\left\{i_{j,1},\ldots,i_{j,\nu_j}\right\}}}^{q-1} \left[\frac{a_{j,1}}{q}+\cdots+\frac{a_{j,n}}{q^n}, \frac{a_{j,1}}{q}+\cdots+\frac{a_{j,n}}{q^n}+\frac{1}{q^n}\right),$$

for some given values of $\nu$, $\mathbf{i}_\nu$, and $\mathbf{a}_\nu$ such that $1 \le i_{j,\nu_j} < \cdots < i_{j,1}$, $j = 1,\ldots,s$, $\nu_j \ge 0$, and

$$\sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j,\alpha)} i_{j,l} \le \beta n - t. \tag{2}$$

We have to show that $J(\mathbf{i}_\nu,\mathbf{a}_\nu)$ contains exactly $q^{m-|\nu|_1}$ points of the digital $(t,\alpha,\beta,n\times m,s)$-net, which we denote by $x_0,\ldots,x_{q^m-1}$. Let $x_h = (x_{h,1},\ldots,x_{h,s})$ and $x_{h,j} = x_{h,j,1}q^{-1} + x_{h,j,2}q^{-2} + \ldots$ be the $q$-adic representation of $x_{h,j}$.

Then for each $0 \le h < q^m$ it follows that $x_h \in J(\mathbf{i}_\nu,\mathbf{a}_\nu)$ if and only if $x_{h,j,k} = a_{j,k}$ for all $k \in \left\{i_{j,1},\ldots,i_{j,\nu_j}\right\}$ and all $j = 1,\ldots,s$. The value of $x_{h,j,k}$ is obtained from the digital construction scheme in the following way: Let $C_1,\ldots,C_s$ denote the generator matrices of the digital $(t,\alpha,\beta,n\times m,s)$-net over $\mathbb{F}_q$. Then $x_{h,j,k} = \varphi^{-1}(\mathbf{c}_{j,k}\mathbf{h})$, where $\mathbf{c}_{j,k}$ denotes the $k$th row of $C_j$. Thus $\mathbf{c}_{j,k}\mathbf{h} = \varphi(x_{h,j,k})$.

Let $C = (\mathbf{c}_{1,i_{1,1}}^\top,\ldots,\mathbf{c}_{1,i_{1,\nu_1}}^\top,\ldots,\mathbf{c}_{s,i_{s,1}}^\top,\ldots,\mathbf{c}_{s,i_{s,\nu_s}}^\top)^\top$ and further we define the vector $\mathbf{b} = (\varphi(a_{1,i_{1,1}}),\ldots,\varphi(a_{1,i_{1,\nu_1}}),\ldots,\varphi(a_{s,i_{s,1}}),\ldots,\varphi(a_{s,i_{s,\nu_s}}))^\top$. Then, by the above, it follows that $x_h \in J(\mathbf{i}_\nu,\mathbf{a}_\nu)$ if and only if $C\mathbf{h} = \mathbf{b}$.

We now investigate how many solutions $\mathbf{h}$ the system of equations $C\mathbf{h} = \mathbf{b}$ has. As (2) is satisfied, Definition 2 implies that the rows of the matrix $C$ are linearly independent. As $C$ has $|\nu|_1$ ($|\nu|_1 \le m$) rows, there are exactly $q^{m-|\nu|_1}$ solutions to this system, and hence $q^{m-|\nu|_1}$ of the $x_0,\ldots,x_{q^m-1}$ fall into $J(\mathbf{i}_\nu,\mathbf{a}_\nu)$, which shows that every digital $(t,\alpha,\beta,n\times m,s)$-net is also a $(t,\alpha,\beta,n,m,s)$-net.

Now we turn to sequences. Let $x_0,x_1,\ldots$ be a digital $(t,\alpha,\beta,\sigma,s)$-sequence over the finite field $\mathbb{F}_q$. Let $k \ge 0$ and $m > t/(\beta\sigma)$. Then the point set $x_{\ell q^m},\ldots,x_{(\ell+1)q^m-1}$ can be obtained from the digital construction scheme with an added digital shift, i.e., there are matrices $C_1,\ldots,C_s \in \mathbb{F}_q^{n\times m}$ and vectors $\mathbf{d}_{j,\ell} = (d_{j,1,\ell},\ldots,d_{j,n,\ell})^\top \in \mathbb{F}_q^n$, $1 \le j \le s$, which depend on $\ell$, such that $x_{h,j,k} = \varphi^{-1}(\mathbf{c}_{j,k}\mathbf{h} + d_{j,k,\ell})$. Thus we have $\mathbf{c}_{j,k}\mathbf{h} = \varphi(x_{h,j,k}) - d_{j,k,\ell} \in \mathbb{F}_q$. For some given generalized elementary interval $J(\mathbf{i}_\nu,\mathbf{a}_\nu)$ we have $x_h \in J(\mathbf{i}_\nu,\mathbf{a}_\nu)$ if and only if $\mathbf{c}_{j,k}\mathbf{h} = \varphi(a_{j,k}) - d_{j,k,\ell}$ for all $k \in \{i_{1,1},\ldots,i_{1,\nu_1},\ldots,i_{s,1},\ldots,i_{s,\nu_s}\}$ and $j = 1,\ldots,s$. Thus the same argument as for nets applies and the result follows.

□

**Definition 6.** Let $q$ be a prime power. Then let $d_q(\alpha,s)$ denote the smallest value of $t$ such that there exists a digital $(t,\alpha,\beta,\sigma,s)$-sequence over the finite field $\mathbb{F}_q$ with $\alpha = \beta\sigma$.

The analogy of Definition 6 for classical digital sequences, i.e. the case $\alpha = 1$, has already appeared in [13], see also [14, Definition 8]. For $\alpha = \beta = \sigma = 1$, i.e. digital $(t,s)$-sequences, it is true that

$$\frac{s}{q-1} - \mathcal{O}(\log s) < d_q(1,s) \leq \frac{c}{\log q} s + 1,$$

for all $s \geq 1$, where $c > 0$ is an absolute constant. The lower bound was shown in [17] and also holds for $(t,s)$-sequences, whereas the upper bound can be found in [13, Theorem 4] and [14, Corollary 1]. Improved results for several special values of $q$ can also be found in [15].

The following theorem now considers the case $\alpha \geq 2$.

**Theorem 7.** *Let $q$ be a prime power. Then for all $s \geq 1$ and $\alpha \geq 2$ we have*

$$s\frac{\alpha(\alpha-1)}{2} - \alpha < d_q(\alpha,s) \leq s\alpha^2 \frac{c}{\log q} + \alpha + \alpha \left\lfloor \frac{s(\alpha-1)}{2} \right\rfloor,$$

*where $c > 0$ is an absolute constant.*

*Proof.* The lower bound is taken from Theorem 5. To prove the upper bound we use Theorem 1 with $d = \alpha$ to obtain a digital $(t,\alpha,1,\alpha,s)$-sequence over $\mathbb{F}_q$ with

$$t = \alpha t' + \alpha \left\lfloor \frac{s(\alpha-1)}{2} \right\rfloor,$$

where $t'$ is the quality parameter of the classical digital $(t',s\alpha)$-sequence upon which the construction is based. From [13, Theorem 4], [14, Corollary 1] we know that there exist digital $(t',s)$-sequences for which $t' \leq \frac{c}{\log q} s + 1$. Upon combining the last two formulae, where we replace $s$ with $\alpha s$ in the last formula as we consider $(t',s\alpha)$-sequences, the result follows.

□

Note that the bounds in Theorem 7 also apply to (non-digital) $(t,\alpha,\beta,\sigma,s)$-sequences with $\alpha = \beta\sigma$ and $t$ value as small as possible.

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