Digital Nets and Sequences

Discrepancy Theory and Quasi-Monte Carlo Integration

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To Jingli and Gisi.

Contents

Preface

The theory of digital nets and sequences has its roots in uniform distribution modulo one and in numerical integration using quasi-Monte Carlo (QMC) rules. The subject can be traced back to several influential works: The notion of uniform distribution goes back to a classical paper by Weyl [263]. The Koskma-Hlawka inequality, which forms the starting point for analysing QMC methods for numerical integration, goes back to Koksma [119] in the one-dimensional case and Hlawka [109] in arbitrary dimension. Explicit constructions of digital sequences were first introduced by Sobol′ [251], followed by Faure [66] and Niederreiter [171]. A general principle of these constructions was introduced by Niederreiter in [170], which now forms one of the essential columns of QMC integration and of this book. These early results are well summarised in [61, 112, 128, 169, 175], where much more information on the history and earlier discoveries can be found.

Since then, numerical integration based on QMC has been developed into a comprehensive theory with many new facets. The introduction of reproducing kernel Hilbert spaces by Hickernell [106] furnished many Koksma-Hlawka type inequalities. The worst-case integration error can be expressed directly in terms of a reproducing kernel, which is a function which, together with a uniquely defined inner product, describes a Hilbert space of functions.

As opposed to earlier believes, QMC methods are now used for numerical integration of functions in hundreds or even thousands of dimensions. The success of this approach has been described by Sloan & Woźniakowski in [247], where the concept of weighted spaces was introduced. These weighted spaces nowadays permeate the literature on high-dimensional numerical integration. The consequence is a weighted Koksma-Hlawka inequality which yields weighted quality measures (called discrepancies) of the quadrature points and the need for constructions of point sets which are of high quality with respect to this new criterion. This leads to computer search algorithms

for suitable quadrature points which were first developed for lattice rules [244, 245] and subsequently extended to polynomial lattice rules [45].

The construction of low-discrepancy point sets and sequences has also undergone dramatic improvements. The constructions of Sobol′ [251], Faure [66], and Niederreiter [171] have been developed into the overarching notion of (digital) (t, m, s)-nets and (t, s)-sequences. The problem of asymptotically optimal constructions in the context of this theory (i.e., which minimise the quality parameter t) have been developed by Niederreiter & Xing in [189, 265], with several subsequent extensions. From a theoretical perspective interesting is the development of a duality theory for digital nets [187], which gives a general framework for the theory of digital nets.

Another development has seen a partial merging of Monte Carlo (MC) methods, where the quadrature points are chosen purely at random, with QMC. The aim here is to introduce a random element into the construction of low-discrepancy points which, on the one hand preserves the distribution properties and is, at the same time, sufficiently random to yield an unbiased estimator (and which has also further useful properties). Such a method, called scrambling, has been introduced by Owen [204], and was first analysed in [205, 207]. As a bonus, one can obtain an improved rate of convergence of $O(N^{-3/2}(\log N)^c)$ (for some $c > 0$) using this randomisation.

The topic of improved rates of convergence was further developed first in [101] for lattice rules, and in [27] for polynomial lattice rules, using a random shift and the tent transformation. This method achieves convergence rates of $O(N^{-2}(\log N)^c)$ (for some $c > 0$). The quadrature points which can be used in this method can be found by computer search.

A general theory of higher order digital nets and sequences has been developed in [36] for periodic functions, and for the general case in [37]. There the convergence rate is of $O(N^{-\alpha}(\log N)^c)$ (for some $c > 0$), with $\alpha > 1$ arbitrarily large for sufficiently smooth functions.

A breakthrough result concerned with the classical problem of finding explicit construction of point sets which achieve the optimal rate of convergence of the L_2-discrepancy has been achieved by Chen & Skriganov [22]. This problem goes back to the lower bound on the L_2-discrepancy by Roth [226].

The aim of this work is to describe these achievements in the area of QMC methods and uniform distribution. The choice and presentation of the topics is naturally biased towards the authors interests and expertise. Another consideration for our choice of topics concerns the monographs already available, many of whom are cited throughout the book.

In order to give a consistent and comprehensive treatment of the subject we use Walsh series analysis throughout the book. In our context these

appeared already in [128, 168] and in the context of analysing digital nets in [131, 146]. Some authors, especially those concerned with the analysis of the mean-square worst-case error of scrambled nets, prefer to use Haar wavelets, which were also used for instance by Sobol′ [250, 251].

In the analysis of scrambled nets, no disadvantage seems to arise from replacing Haar functions with Walsh functions. The locality of Haar functions is offset by the locality of the Walsh-Dirichlet kernel. As illustration, Owen's description of a nested ANOVA decomposition [205] can also be neatly described using the Walsh-Dirichlet kernel, see Section 13.2. The place where it turned out that Walsh functions are of considerable advantage is in Chapter 14. The Walsh coefficients of smooth functions exhibit a certain decay which is an essential ingredient in the theory on higher order digital nets and sequences. This property is not shared in the same manner by the Haar coefficients of smooth functions. Furthermore, also the construction of point sets with optimal L_2-discrepancy has its origin in the Walsh series expansion of the characteristic function $\chi_{[0,x)}$. This makes Walsh functions more suitable for our endeavour than Haar functions. However, this shall not mean that this is the case in all situations, in future work authors should consider such a choice on a case by case basis.

The aim of the book is to give an introduction to the topics described above as well as some others. Parts of the theory which already appeared elsewhere are repeated here to make the monograph as self-contained as possible. This effort is complemented by two appendices, one on Walsh functions and one on algebraic function fields. The latter one are the underlying basis for the constructions of digital nets and sequences by Niederreiter, Xing, and Özbudak described in Chapter 8.

The text is aimed at undergraduate students in Mathematics. The exercises at the end of each chapter make it suitable for an undergraduate or graduate course on the topic of this book or parts thereof. Such a course may be useful for students in science, engineering, or finance, where QMC methods find their applications. We also hope that it may prove useful for our colleagues as reference book and inspiration for future work. We hope for a similar advancement of the area in the next decades as we have seen in the past.

Acknowledgements

The beginning of this book goes back many years now to a hand-written manuscript by Gerhard Larcher, which was the basis for the first authors master's thesis under Gerhard's supervision and which now forms the main

part of Chapters 4 and 5. This manuscript was in fact the first comprehensive introduction to the topic for the authors. For this and other contributions we are immensely grateful. Thank you Gerhard!

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1
Introduction

In this introductory chapter we review some current methods of numerical integration to put the subsequent chapters into a wider context. This serves as a motivation for later investigations.

The problem of numerical integration occurs in applications from physics, chemistry, finance, biology, computer graphics, and others, where one has to compute some integral (for instance an expectation value) which cannot be done analytically. Hence one has to resort to numerical methods in this case. We shall in the following consider only the standardised problem of approximating an integral of the form

$$\int_{[0,1]^s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.$$

The books of Fox [79], Tezuka [254], Glasserman [83], and Lemieux [152] and the surveys of Keller [118] and L'Ecuyer [149] deal more directly with questions arising from applications.

1.1 The one-dimensional case

Let us consider the case $s = 1$ first. Let $f : [0,1] \to \mathbb{R}$ be a Riemann integrable function. We proceed now as follows. Take a sample of N points $x_0, \ldots, x_{N-1}$ in the interval $[0,1)$ and calculate the average function value at those points, i.e.,

$$\frac{1}{N}\sum_{n=0}^{N-1} f(x_n).$$

As approximation to the integral we use the value

$$\text{length of the interval } \times \text{ average function value},$$

that is, we approximate the integral of f by

$$\int_0^1 f(x)\,\mathrm{d}x \approx \frac{1}{N}\sum_{n=0}^{N-1} f(x_n).$$

The question arises how large the approximation error is using this method, i.e., how large is the value

$$\left|\int_0^1 f(x)\,\mathrm{d}x - \frac{1}{N}\sum_{n=0}^{N-1} f(x_n)\right|?$$

Intuitively we expect the integration error to depend on two quantities, namely,

- on the quadrature points $x_0, \dots, x_{N-1} \in [0,1)$, and
- on the function f.

Let us consider those two points in turn. The quadrature points should have no big gaps in between otherwise large portions of the function are not considered in the approximation. Hence $\{x_0, \dots, x_{N-1}\}$ should be well distributed in $[0,1)$. For instance, assume we want to integrate the function $f : [0,1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 0 & \text{if } x \le 1/2, \\ 1 & \text{if } x > 1/2. \end{cases}$$

If all the points $x_0, \dots, x_{N-1}$ are in the interval $[0, 1/2]$, i.e., the points are not well distributed in $[0,1)$, then we obtain

$$\frac{1}{N}\sum_{n=0}^{N-1} f(x_n) = 0$$

as an approximation to the integral

$$\int_0^1 f(x)\,\mathrm{d}x = 1/2,$$

see Figure 1.1.

Hence we obtain an integration error

$$\left|\frac{1}{N}\sum_{n=0}^{N-1} f(x_n) - \int_0^1 f(x)\,\mathrm{d}x\right| = \frac{1}{2}.$$

The error depends of course also strongly on the integrand f itself, and in particular on the smoothness and some norm of the integrand f, which in some sense measures how strongly f varies. For instance, constant functions

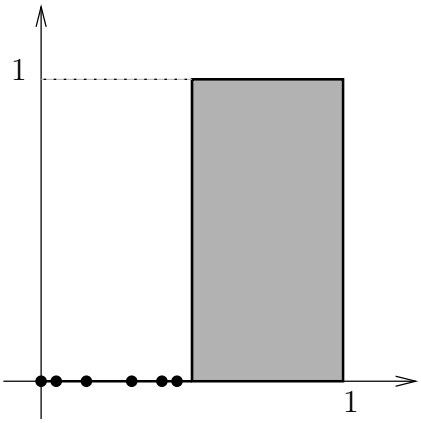

Figure 1.1 Example of badly distributed quadrature points.

are always integrated exactly with our method. On the other hand, assume we have an integrand f which varies strongly, like the function $f(x) = 1+\cos(2\pi kx)$ in Figure 1.2 for some large value of k. If we choose for $N = k$

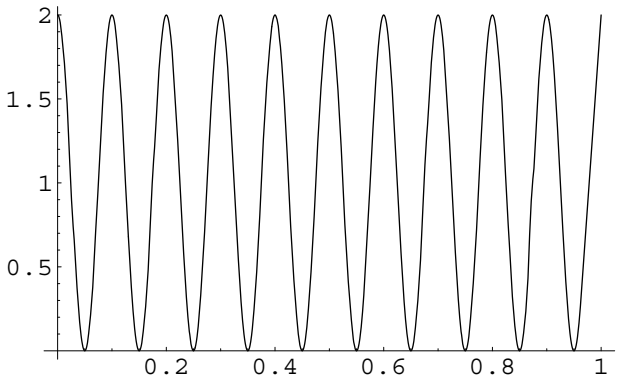

Figure 1.2 Example of the strongly varying function $f(x) = 1 + \cos(2\pi kx)$ with $k = 10$.

the points $x_0, \ldots, x_{N-1}$ as $x_n = (2n + 1)/(2N)$ for $0 \leq n < N$, then one may say that they are "well" distributed in $[0, 1)$, but we still obtain a large integration error. Indeed we have

$$\int_0^1 f(x)\,\mathrm{d}x = 1,$$

but

$$\frac{1}{N}\sum_{n=0}^{N-1} f(x_n) = 0.$$

Hence again we obtain a large integration error

$$\left|\frac{1}{N}\sum_{n=0}^{N-1} f(x_n) - \int_0^1 f(x)\,\mathrm{d}x\right| = 1.$$

Remark 1.1 We see later in Chapter 2 that one can indeed bound the integration error by a product of a quantity which measures the distribution properties of the points $x_0, \ldots, x_{N-1}$ and a quantity which measures how strongly the integrand f varies.

1.2 The general case

Now let us consider the case where $s \in \mathbb{N}$. Let $f : [0,1]^s \to \mathbb{R}$ be a, say, Riemann integrable function. We want to approximate the value of the integral

$$\int_0^1 \cdots \int_0^1 f(x_1, \ldots, x_s)\,\mathrm{d}x_1 \cdots \mathrm{d}x_s = \int_{[0,1]^s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.$$

For this purpose we proceed as in the case $s = 1$, i.e., we choose quadrature points $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1} \in [0,1)^s$ and approximate the integral via the average function value of f at those N points, i.e.,

$$\int_{[0,1]^s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \approx \frac{1}{N}\sum_{n=0}^{N-1} f(\boldsymbol{x}_n).$$

Again we want to estimate the absolute value of the integration error

$$\left|\int_{[0,1]^s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \frac{1}{N}\sum_{n=0}^{N-1} f(\boldsymbol{x}_n)\right|.$$

Now the question arises how we should choose the quadrature points $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}$. Considering the case $s = 1$, a solution which suggests itself for $s > 1$ would be to choose the points on a centred regular lattice. For $s = 1$ we would choose, as above, the points $x_n = \frac{2n+1}{2N}$ for $0 \le n < N$. In general, for $m \in \mathbb{N}$, $m \ge 2$, the *centred regular lattice* Γ_m^{c} is given by the points

$$\boldsymbol{x}_{\boldsymbol{k}} = \left(\frac{2k_1+1}{2m}, \ldots, \frac{2k_s+1}{2m}\right) \tag{1.1}$$

for all $\boldsymbol{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ with $|\boldsymbol{k}|_\infty := \max_{1 \le i \le s} |k_i| < m$ (hence we

have $N = m^s$ points). An example of a centred regular lattice is shown in Figure 1.3.

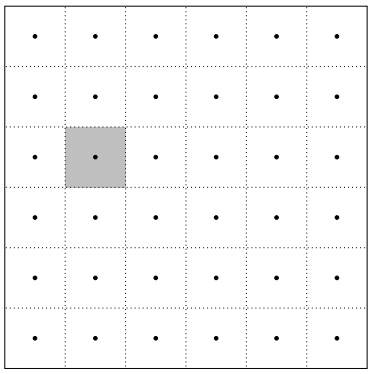

Figure 1.3 Centred regular lattice Γ_6^{c} in $[0,1)^2$, i.e., $s = 2$ and $m = 6$.

As mentioned above, we need to make some smoothness assumptions on the integrand f. In the following we therefore assume that the integrand f is continuous. In this case we can introduce the following concept as a measure of how much the function f varies.

Definition 1.2 For a continuous function $f : [0,1]^s \to \mathbb{R}$, the *modulus of continuity* is given by

$$M_f(\delta) := \sup_{\substack{\boldsymbol{x},\boldsymbol{y}\in[0,1]^s \\ |\boldsymbol{x}-\boldsymbol{y}|_\infty \le \delta}} |f(\boldsymbol{x}) - f(\boldsymbol{y})| \quad \text{for} \quad \delta \ge 0,$$

where $|\cdot|_\infty$ is the maximum norm, i.e., for $\boldsymbol{x} = (x_1, \ldots, x_s)$ we set $|\boldsymbol{x}|_\infty := \max_{1\le i\le s} |x_i|$.

If we assume that the function f is uniformly continuous on $[0,1]^s$, then we have $\lim_{\delta\to 0^+} M_f(\delta) = 0$. Note that for any function f its modulus M_f is nondecreasing and subadditive. Recall that a function f is nondecreasing if $f(x) \le f(y)$ for all $x \le y$, and that a function f is subadditive if $f(x+y) \le f(x) + f(y)$ for all x, y in the domain of f.

Furthermore, for nonconstant functions f the smallest possible order of M_f is $M_f(\delta) = O(\delta)$ as $\delta \to 0^+$. Recall that we say $h(x) = O(g(x))$ as $x \to 0$ if and only if there exist positive real numbers δ and C such that $|h(x)| \le C|g(x)|$ for $|x| < \delta$.

For $\boldsymbol{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ with $|\boldsymbol{k}|_\infty < m$ let $Q_{\boldsymbol{k}} = \prod_{i=1}^s [k_i/m, (k_i+1)/m)$. Then each point of the centred regular lattice (1.1) is contained in exactly one interval $Q_{\boldsymbol{k}}$, namely the point $\boldsymbol{x}_{\boldsymbol{k}}$ (see again Figure 1.3).

Let now $f : [0,1]^s \to \mathbb{R}$ be a continuous function and let $\boldsymbol{x}_{\boldsymbol{k}}$ for $\boldsymbol{k} \in \mathbb{N}_0^s$ with $|\boldsymbol{k}|_\infty < m$ be the points of a centred regular lattice. Then we have

$$\begin{aligned}\left|\int_{[0,1]^s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \frac{1}{m^s}\sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^s\\ |\boldsymbol{k}|_\infty<m}} f(\boldsymbol{x}_{\boldsymbol{k}})\right| &= \left|\sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^s\\ |\boldsymbol{k}|_\infty<m}} \int_{Q_{\boldsymbol{k}}} f(\boldsymbol{x}) - f(\boldsymbol{x}_{\boldsymbol{k}})\,\mathrm{d}\boldsymbol{x}\right| \\ &\le \sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^s\\ |\boldsymbol{k}|_\infty<m}} \int_{Q_{\boldsymbol{k}}} M_f(|\boldsymbol{x}-\boldsymbol{x}_k|_\infty)\,\mathrm{d}\boldsymbol{x} \\ &\le m^s \int_{B(\frac{1}{2m})} M_f(|\boldsymbol{x}|_\infty)\,\mathrm{d}\boldsymbol{x}, \qquad (1.2)\end{aligned}$$

where $B(\varepsilon) := \{\boldsymbol{x} \in \mathbb{R}^s : |\boldsymbol{x}|_\infty \le \varepsilon\}$.

Assume that the function f is in addition Lipschitz continuous (for example it suffices if f has partial derivatives), i.e., there is a real number $C_f > 0$ such that

$$M_f(\delta) \le C_f\delta \quad \text{for all } \delta > 0.$$

Then, using (1.2), we have

$$\begin{aligned}\left|\int_{[0,1]^s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \frac{1}{m^s}\sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^s\\ |\boldsymbol{k}|_\infty<m}} f(\boldsymbol{x}_{\boldsymbol{k}})\right| &\le m^s \int_{B(\frac{1}{2m})} C_f|\boldsymbol{x}|_\infty\,\mathrm{d}\boldsymbol{x} \\ &\le \frac{C_f}{2m} = \frac{C_f}{2N^{1/s}}, \qquad (1.3)\end{aligned}$$

where the last inequality can be obtained by estimating $|\boldsymbol{x}|_\infty \le \frac{1}{2m}$.

This result cannot be improved significantly for uniformly continuous functions. Before we show the corresponding result, let us give some examples. For instance, choose $s = 1$ and consider the function $f(x) = \frac{c}{2N}(1 + \cos(2\pi N x))$ for some constant $c > 0$ (see Figure 1.4). Notice that $f'(x) = -c\pi\sin(2\pi N x)$, hence the Lipschitz constant is $C_f = \sup_{0\le x\le 1}|f'(x)| = c\pi$ and the modulus of continuity satisfies $M_f(\delta) \le c\pi\delta$ for all $\delta > 0$. Thus, as opposed to the function itself, the Lipschitz constant and the modulus of continuity do not depend on N. If we consider the Lipschitz constant or the modulus of continuity of f as a measure of how strongly f varies, then this measure does not depend on N. Hence we have a family of functions which all vary equally strongly. Let us now consider the integration errors of these functions.

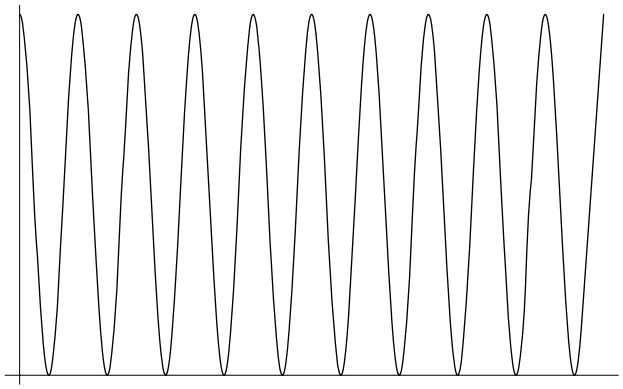

Figure 1.4 The function $f(x) = \frac{c}{2N}(1 + \cos(2\pi N x))$ for $N = 10$.

We have

$$\frac{1}{N}\sum_{n=0}^{N-1} f\left(\frac{2n+1}{2N}\right) = 0 \quad \text{and} \quad \int_0^1 f(x)\,\mathrm{d}x = \frac{c}{2N},$$

and hence

$$\left|\int_0^1 f(x)\,\mathrm{d}x - \frac{1}{N}\sum_{n=0}^{N-1} f\left(\frac{2n+1}{2N}\right)\right| = \frac{c}{2N}.$$

A convergence of $O(N^{-1})$ is reasonable in many practical applications, which makes the quadrature method a useful tool in dimension $s = 1$.

Consider now the case $s > 1$. Choose a function

$$g(x_1, x_2, \ldots, x_s) = \frac{c}{2m}(1 + \cos(2\pi m x_1)),$$

see Figure 1.5. Again, the functions g vary equally strongly for each m.

Then we have

$$\int_0^1 \cdots \int_0^1 g(x_1, \ldots, x_s)\,\mathrm{d}x_1 \cdots \mathrm{d}x_s = \frac{c}{2m} = \frac{c}{2N^{1/s}}$$

and

$$\frac{1}{m^s} \sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^s \\ |\boldsymbol{k}|_\infty < m}} g(\boldsymbol{x}_{\boldsymbol{k}}) = 0.$$

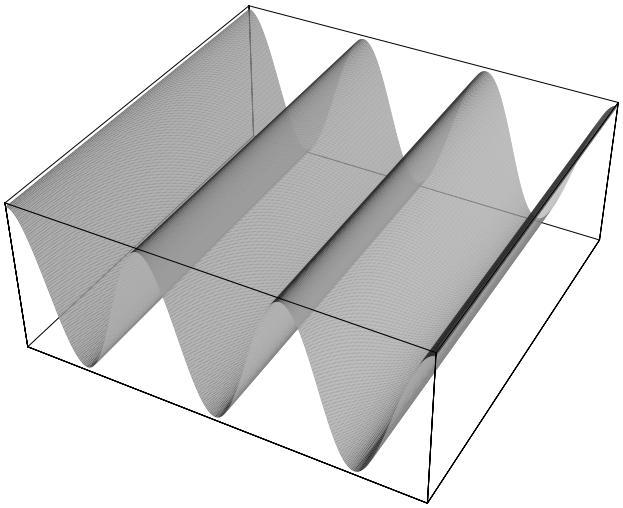

Figure 1.5 The function $g(x_1, x_2) = \frac{c}{2m}(1 + \cos(2\pi m x_1))$ for $m = 3$.

Hence we obtain an integration error of

$$\left| \int_{[0,1]^s} f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} - \frac{1}{m^s} \sum_{\substack{\boldsymbol{k} \in \mathbb{N}_0^s \\ |\boldsymbol{k}|_\infty < m}} g(\boldsymbol{x}_{\boldsymbol{k}}) \right| = \frac{c}{2N^{1/s}}.$$

Motivated by the above examples we show the following unpublished result due to G. Larcher. In the following we call a uniformly continuous function $M : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, where $\mathbb{R}_0^+ = \{x \in \mathbb{R} : x \geq 0\}$, which is nondecreasing, subadditive, and for which we have $\lim_{\delta \to 0^+} M(\delta) = 0$ a *modulus*.

Theorem 1.3 *For any modulus M and any $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}$ in $[0,1)^s$, there is a uniformly continuous function $f : [0,1]^s \to \mathbb{R}$ with modulus of continuity $M_f \leq M$, such that*

$$\left| \int_{[0,1]^s} f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} - \frac{1}{N} \sum_{n=0}^{N-1} f(\boldsymbol{x}_n) \right| \geq N \int_{B\left(\frac{1}{2N^{1/s}}\right)} M(|\boldsymbol{x}|_\infty) \, \mathrm{d}\boldsymbol{x}.$$

Proof Consider the Voronoi diagram $V_0, \ldots, V_{N-1}$ of $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}$ with respect to the maximum norm, i.e.,

$$V_n = \{\boldsymbol{x} \in [0,1]^s \, : \, |\boldsymbol{x} - \boldsymbol{x}_n|_\infty = \min_{0 \leq j < N} |\boldsymbol{x} - \boldsymbol{x}_j|_\infty\}$$

for $0 \leq n < N$, and define $f : [0,1]^s \to \mathbb{R}$ by $f(\boldsymbol{x}) := M(|\boldsymbol{x} - \boldsymbol{x}_n|_\infty)$ for

$\boldsymbol{x} \in V_n$. Then f is uniformly continuous since M is continuous, $V_0, \dots, V_{N-1}$ is a Voronoi diagram, and f is defined on a compact domain.

We show that $M_f \le M$. Let $\boldsymbol{x}, \boldsymbol{y} \in [0,1]^s$ and assume that $f(\boldsymbol{x}) > f(\boldsymbol{y})$. If $\boldsymbol{x}, \boldsymbol{y}$ are in the same Voronoi cell, say V_n, then we have

$$\begin{aligned} |f(\boldsymbol{x}) - f(\boldsymbol{y})| &= M(|\boldsymbol{x} - \boldsymbol{x}_n|_\infty) - M(|\boldsymbol{y} - \boldsymbol{x}_n|_\infty) \\ &\le M(\max(|\boldsymbol{x} - \boldsymbol{x}_n|_\infty - |\boldsymbol{y} - \boldsymbol{x}_n|_\infty, 0)) \\ &\le M(|\boldsymbol{x} - \boldsymbol{y}|_\infty), \end{aligned}$$

where we used that M is subadditive and nondecreasing. If $\boldsymbol{x}, \boldsymbol{y}$ are not in the same Voroni cell, say $\boldsymbol{x} \in V_n$ and $\boldsymbol{y} \in V_k$ with $n \ne k$, then we have

$$\begin{aligned} |f(\boldsymbol{x}) - f(\boldsymbol{y})| &= M(|\boldsymbol{x} - \boldsymbol{x}_n|_\infty) - M(|\boldsymbol{y} - \boldsymbol{x}_k|_\infty) \\ &\le M(|\boldsymbol{x} - \boldsymbol{x}_k|_\infty) - M(|\boldsymbol{y} - \boldsymbol{x}_k|_\infty) \\ &\le M(\max(|\boldsymbol{x} - \boldsymbol{x}_k|_\infty - |\boldsymbol{y} - \boldsymbol{x}_k|_\infty, 0)) \\ &\le M(|\boldsymbol{x} - \boldsymbol{y}|_\infty), \end{aligned}$$

where we again used that M is subadditive and nondecreasing. Hence we have $M_f \le M$.

It remains to show the lower bound on the integration error. We have

$$\left| \int_{[0,1]^s} f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} - \frac{1}{N} \sum_{n=0}^{N-1} f(\boldsymbol{x}_n) \right| = \sum_{n=0}^{N-1} \int_{V_n} M(|\boldsymbol{x} - \boldsymbol{x}_n|_\infty) \, \mathrm{d}\boldsymbol{x}. \tag{1.4}$$

Let $W_n := \left\{ \boldsymbol{x} \in [0,1]^s \,:\, |\boldsymbol{x} - \boldsymbol{x}_n|_\infty \le 1/(2N^{1/s}) \right\}$. Then we have

$$\begin{aligned} &\sum_{n=0}^{N-1} \int_{V_n} M(|\boldsymbol{x} - \boldsymbol{x}_n|_\infty) \, \mathrm{d}\boldsymbol{x} \\ &= \sum_{n=0}^{N-1} \left(\int_{V_n \cap W_n} M(|\boldsymbol{x} - \boldsymbol{x}_n|_\infty) \, \mathrm{d}\boldsymbol{x} + \int_{V_n \setminus W_n} M(|\boldsymbol{x} - \boldsymbol{x}_n|_\infty) \, \mathrm{d}\boldsymbol{x} \right). \end{aligned} \tag{1.5}$$

Let $\boldsymbol{y} \in V_n \setminus W_n$ for some n and let $\boldsymbol{x} \in W_k \setminus V_k$ for some k. Then we have $|\boldsymbol{y} - \boldsymbol{x}_n|_\infty > 1/(2N^{1/s})$ and $|\boldsymbol{x} - \boldsymbol{x}_k| \le 1/(2N^{1/s})$. Since M, by definition, is nondecreasing it follows that $M(|\boldsymbol{y} - \boldsymbol{x}_n|_\infty) \ge M(|\boldsymbol{x} - \boldsymbol{x}_k|_\infty)$.

We also have $\sum_{n=0}^{N-1} \lambda_s(V_n) = 1$ and $\sum_{n=0}^{N-1} \lambda_s(W_n) \le 1$, where λ_s is the s-dimensional Lebesgue measure. Hence we have

$$\begin{aligned} 0 &\le \sum_{n=0}^{N-1} \lambda_s(V_n) - \sum_{n=0}^{N-1} \lambda_s(W_n) \\ &= \sum_{n=0}^{N-1} [\lambda_s(V_n \setminus W_n) + \lambda_s(V_n \cap W_n)] - \sum_{n=0}^{N-1} [\lambda_s(W_n \setminus V_n) + \lambda_s(V_n \cap W_n)] \end{aligned}$$

$$= \sum_{n=0}^{N-1} \lambda_s(V_n \setminus W_n) - \sum_{n=0}^{N-1} \lambda_s(W_n \setminus V_n),$$

from which it follows that $\sum_{n=0}^{N-1} \lambda_s(W_n \setminus V_n) \le \sum_{n=0}^{N-1} \lambda_s(V_n \setminus W_n)$. From these considerations it follows that

$$\sum_{n=0}^{N-1} \int_{V_n \setminus W_n} M(|\boldsymbol{x} - \boldsymbol{x}_n|_\infty)\,\mathrm{d}\boldsymbol{x} \ge \sum_{n=0}^{N-1} \int_{W_n \setminus V_n} M(|\boldsymbol{x} - \boldsymbol{x}_n|_\infty)\,\mathrm{d}\boldsymbol{x}.$$

Inserting this inequality in (1.5), then we obtain

$$\begin{aligned}
&\sum_{n=0}^{N-1} \int_{V_n} M(|\boldsymbol{x} - \boldsymbol{x}_n|_\infty)\,\mathrm{d}\boldsymbol{x} \\
&\ge \sum_{n=0}^{N-1} \left(\int_{V_n \cap W_n} M(|\boldsymbol{x} - \boldsymbol{x}_n|_\infty)\,\mathrm{d}\boldsymbol{x} + \int_{W_n \setminus V_n} M(|\boldsymbol{x} - \boldsymbol{x}_n|_\infty)\,\mathrm{d}\boldsymbol{x} \right) \\
&= \sum_{n=0}^{N-1} \int_{W_n} M(|\boldsymbol{x} - \boldsymbol{x}_n|_\infty)\,\mathrm{d}\boldsymbol{x} \\
&= N \int_{B\left(\frac{1}{2N^{1/s}}\right)} M(|\boldsymbol{x}|_\infty)\,\mathrm{d}\boldsymbol{x}.
\end{aligned}$$

Now the result follows by (1.4). □

Combining Theorem 1.3 with (1.2) we obtain the following result from G. Larcher, which states that the centred regular lattice yields the smallest possible integration error for the class of uniformly continuous functions with a given modulus of continuity.

Corollary 1.4 *Let $N = m^s$ and let M be any modulus. Then we have*

$$\inf_{\mathcal{P}} \sup_{f} \left| \int_{[0,1]^s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \frac{1}{N} \sum_{n=0}^{N-1} f(\boldsymbol{x}_n) \right| = N \int_{B\left(\frac{1}{2N^{1/s}}\right)} M(|\boldsymbol{x}|_\infty)\,\mathrm{d}\boldsymbol{x},$$

where the infimum is extended over all point sets $\mathcal{P}$ consisting of N points in $[0,1)^s$ and the supremum is extended over all uniformly continuous functions $f : [0,1]^s \to \mathbb{R}$ with modulus of continuity $M_f = M$. Moreover, the infimum is attained by the centred regular lattice.

The problem in the upper bounds (1.2) and (1.3) respectively is that the integration error depends strongly on the dimension s. For large s the convergence of $N^{-1/s}$ to 0 is very slow as $N \to \infty$. This phenomenon is often called the *curse of dimensionality*. The question arises whether one

can choose "better" quadrature points $\boldsymbol{x}_0, \dots, \boldsymbol{x}_{N-1}$, i.e., for which the integration error depends only weakly (or not at all) on the dimension s. The question can be answered in the affirmative, which can be seen by the following consideration.

Assume we want to approximate the integral of a function $f : [0,1]^s \to \mathbb{R}$ by

$$\frac{1}{N} \sum_{n=0}^{N-1} f(\boldsymbol{x}_n),$$

where $\boldsymbol{x}_0, \dots, \boldsymbol{x}_{N-1} \in [0,1)^s$. Then one can ask how large the integration error

$$\left| \int_{[0,1]^s} f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} - \frac{1}{N} \sum_{n=0}^{N-1} f(\boldsymbol{x}_n) \right| =: R_{N,f}(\boldsymbol{x}_0, \dots, \boldsymbol{x}_{N-1})$$

is on average. That is, if one chooses the quadrature points $\boldsymbol{x}_0, \dots, \boldsymbol{x}_{N-1} \in [0,1)^s$ uniformly distributed and i.i.d., how large is $R_{N,f}$ on average, i.e., what is the expectation value of $R_{N,f}$?

Let $f : [0,1]^s \to \mathbb{R}$ be a square integrable function, i.e., $f \in L_2([0,1]^s)$. In the following we calculate the expectation value of $R_{N,f}^2$, i.e., $\mathbb{E}[R_{N,f}^2]$ and then use the inequality

$$\mathbb{E}[R_{N,f}] \leq \sqrt{\mathbb{E}[R_{N,f}^2]}.$$

Let $g(\boldsymbol{x}) := f(\boldsymbol{x}) - \int_{[0,1]^s} f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}$. Then we have

$$\int_{[0,1]^s} g(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} = 0. \tag{1.6}$$

Now we have

$$\begin{aligned}
&\left(\frac{1}{N} \sum_{n=0}^{N-1} f(\boldsymbol{x}_n) - \int_{[0,1]^s} f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \right)^2 \\
&= \left(\frac{1}{N} \sum_{n=0}^{N-1} g(\boldsymbol{x}_n) \right)^2 \\
&= \frac{1}{N^2} \sum_{n=0}^{N-1} g^2(\boldsymbol{x}_n) + \frac{2}{N^2} \sum_{0 \leq m < n < N} g(\boldsymbol{x}_m) g(\boldsymbol{x}_m).
\end{aligned}$$

Hence

$$\mathbb{E}[R_{N,f}^2] = \frac{1}{N^2} \sum_{n=0}^{N-1} \int_{[0,1]^s} \cdots \int_{[0,1]^s} g^2(\boldsymbol{x}_n) \, \mathrm{d}\boldsymbol{x}_0 \cdots \mathrm{d}\boldsymbol{x}_{N-1}$$

$$+\frac{2}{N^2}\sum_{0\le m<n<N}\int_{[0,1]^s}\cdots\int_{[0,1]^s} g(\boldsymbol{x}_m)g(\boldsymbol{x}_n)\,\mathrm{d}\boldsymbol{x}_0\cdots\mathrm{d}\boldsymbol{x}_{N-1}$$
$$=: \Sigma_1+\Sigma_2.$$

We consider Σ_2. For any $0\le m<n<N$, (1.6) implies that

$$\int_{[0,1]^s}\cdots\int_{[0,1]^s} g(\boldsymbol{x}_m)g(\boldsymbol{x}_n)\,\mathrm{d}\boldsymbol{x}_0\cdots\mathrm{d}\boldsymbol{x}_{N-1}$$
$$=\int_{[0,1]^s} g(\boldsymbol{x}_m)\,\mathrm{d}\boldsymbol{x}_m\int_{[0,1]^s} g(\boldsymbol{x}_n)\,\mathrm{d}\boldsymbol{x}_n=0.$$

Hence $\Sigma_2=0$ and therefore $\mathbb{E}[R_{N,f}^2]=\Sigma_1$. Further for every $0\le n<N$ we have

$$\int_{[0,1]^s}\cdots\int_{[0,1]^s} g(\boldsymbol{x}_n)^2\,\mathrm{d}\boldsymbol{x}_0\cdots\mathrm{d}\boldsymbol{x}_{N-1}=\int_{[0,1]^s} g^2(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$$
$$=\int_{[0,1]^s}\left(f(\boldsymbol{x})-\int_{[0,1]^s} f(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\right)^2\mathrm{d}\boldsymbol{x}.$$

Hence we have the following theorem.

Theorem 1.5 *Let $f\in L_2([0,1]^s)$. Then for any $N\in\mathbb{N}$ we have*

$$\mathbb{E}[R_{N,f}^2]=\frac{1}{N}\int_{[0,1]^s}\left(f(\boldsymbol{x})-\int_{[0,1]^s} f(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\right)^2\mathrm{d}\boldsymbol{x}=\frac{\sigma^2(f)}{N},$$

where we set $\sigma^2(f):=\int_{[0,1]^s}\left(f(\boldsymbol{x})-\int_{[0,1]^s} f(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\right)^2\mathrm{d}\boldsymbol{x}$.

Theorem 1.5 can now be understood in the following way. The absolute value of the integration error is, on average, bounded by $\sigma(f)/\sqrt{N}$, where $\sigma(f)=\sqrt{\sigma^2(f)}$ is the standard deviation of f. Note that the integration error does not depend on the dimension s (although for some functions $\sigma(f)$ may depend on s). We have $N^{-1/2}<N^{-1/s}$ for $s>2$. Hence, roughly speaking, for $s>2$ it is on average better to use random points for the approximation of the integral of f than using the centred regular grid (f does not even have to be continuous if one chooses random samples). This method of using random sample points $\boldsymbol{x}_0,\dots,\boldsymbol{x}_{N-1}$ is called *Monte Carlo (MC) method*.

Nevertheless the MC method also has some disadvantages:

- The error bound is only probabilistic, that is, in any one instance one cannot be sure of the integration error. However, further probabilistic

information is obtained from the central limit theorem, which states (see [175]) that, if $0 < \sigma(f) < \infty$, then

$$\lim_{N\to\infty} \text{Prob}\left[R_{N,f}(\boldsymbol{x}_0,\ldots,\boldsymbol{x}_{N-1}) \le \frac{c\sigma(f)}{\sqrt{N}}\right] = \sqrt{\frac{2}{\pi}}\int_0^c \mathrm{e}^{-t^2/2}\,\mathrm{d}t,$$

for any $c > 0$, where Prob[·] is the infinite-dimensional Lebesgue measure λ_∞ of all sequences $\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots$ of elements of $[0,1)^s$ that have the property indicated between the brackets.

- A second problem is that the generation of random samples is difficult. This problem is a topic on its own. For more information in this direction we refer to the books of Lemieux [152], of Niederreiter [175], or to the overview article of L'Ecuyer & Hellekalek [150].
- The convergence rate of $O(N^{-1/2})$ is for some applications too slow and it does not reflect some regularity of the integrand.

For more information concerning the MC method we refer to the books of Niederreiter [175], of Lemieux [152] or of Glasserman [83]. The later one deals with the application of MC for financial problems.

The aim is now to find *deterministic* constructions of quadrature points which are at least as good as the average. This method is called *quasi-Monte Carlo (QMC) method* as opposed to MC, where one uses randomly chosen quadrature points. In the deterministic case we hence need quadrature points which are in some sense "well" distributed in $[0,1)^s$. We consider this problem in the next two chapters. There we also specify the space of integrands first, since this also determines what the correct distribution properties of the quadrature points should be. Chapter 3 motivates the distribution properties of the quadrature points from a geometrical point of view and presents some classical constructions of "good" quadrature points.

Exercises

1.1 Define a modulus M by $M(\delta) = \delta$ for all $\delta \ge 0$. Find a function $f : [0,1] \to \mathbb{R}$ which has modulus of continuity $M_f \le M$. Verify the lower bound on the integration error of Theorem 1.3 for this function.

1.2 A well known measure of how strongly a function $f : [0,1] \to \mathbb{R}$ varies is the so-called *total variation* $V(f)$. For functions whose first derivative f' is continuous it is known that the total variation can be computed by

$$V(f) = \int_0^1 |f'(x)|\,\mathrm{d}x.$$

(Note that this is a semi-norm of f which should be compared to the norm in the Koksma-Hlawka inequality, which is presented in the following chapter as Proposition 2.18.)

Compute the total variation of the function $f(x) = \frac{c}{2N}(1+\cos(2\pi N x))$. *Remark:* Observe that the total variation of this function is independent of N.

1.3 Assume that the function $f : [0,1]^s \to \mathbb{R}$ satisfies a Hölder condition, i.e., $|f(\boldsymbol{x}) - f(\boldsymbol{y})| \le C_f |\boldsymbol{x} - \boldsymbol{y}|_\infty^\lambda$ for some constant $C_f > 0$ which only depends on f and $0 < \lambda \le 1$. Show that then

$$\left| \int_{[0,1]^s} f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} - \frac{1}{m^s} \sum_{\substack{\boldsymbol{k} \in \mathbb{N}_0^s \\ |\boldsymbol{k}|_\infty < m}} f(\boldsymbol{x}_{\boldsymbol{k}}) \right| \le \frac{C_f}{2^\lambda m^\lambda},$$

where $\boldsymbol{x}_{\boldsymbol{k}}$ with $\boldsymbol{k} \in \mathbb{N}_0^s$ and $|\boldsymbol{k}|_\infty < m$ is a centred regular lattice.

1.4 Let $m^s < N < (m+1)^s$, and in particular $m^l (m+1)^{s-l} \le N < m^{l-1}(m+1)^{s-l+1}$ for some $1 \le l \le s$, say $N = m^l (m+1)^{s-l} + k$ with some $0 \le k < m^{l-1}(m+1)^{s-l}$. Consider $m^{l-1}(m+1)^{s-l}$ intervals

$$\prod_{i=1}^{l-1} \left[\frac{a_i}{m}, \frac{a_i+1}{m} \right) \times \prod_{i=l}^{s-1} \left[\frac{b_i}{m+1}, \frac{b_i+1}{m+1} \right) \times [0,1)$$

with $0 \le a_i < m$ and $0 \le b_i < m+1$. For $m^{l-1}(m+1)^{s-l} - k$ of these intervals divide the last coordinate into m equal parts and for the remaining k intervals divide the last coordinate into $m+1$ equal parts. This gives N boxes. Take the N mid points of these boxes. This gives a *centred quasi-regular lattice*. See Figure 1.6 for an example.

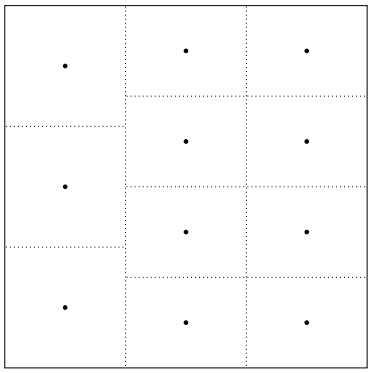

Figure 1.6 Centred quasi-regular lattice in $[0,1)^2$ with $s = 2$ $N = 11$, $m = 3$, $l = 2$, and $k = 2$.

Let $f : [0,1]^s \to \mathbb{R}$ be continuous with modulus of continuity M_f and let $r(f) := \sup_{\boldsymbol{x}\in[0,1]^s} f(\boldsymbol{x})$. Show that for a centred quasi-regular lattice $\boldsymbol{x}_0, \dots, \boldsymbol{x}_{N-1}$ with $m^s < N < (m+1)^s$ we have

$$\left| \int_{[0,1]^s} f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} - \frac{1}{N} \sum_{n=0}^{N-1} f(\boldsymbol{x}_n) \right| \le N \int_{B(\frac{1}{2m})} \omega(|\boldsymbol{x}|_\infty) \, \mathrm{d}\boldsymbol{x} + r(f) \frac{1}{N^{1/s}}.$$

1.5 Let $f : [0,1]^s \to \mathbb{R}$ be Lipschitz continuous. Show that for a quasi-regular lattice $\boldsymbol{x}_0, \dots, \boldsymbol{x}_{N-1}$ with $m^s < N < (m+1)^s$ we have

$$\left| \int_{[0,1]^s} f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} - \frac{1}{N} \sum_{n=0}^{N-1} f(\boldsymbol{x}_n) \right| = O(N^{-1/s}).$$

1.6 For a Borel set $E \subseteq [0,1]^s$ we say a point set $\mathcal{P} = \{\boldsymbol{x}_0, \dots, \boldsymbol{x}_{N-1}\}$ in $[0,1]^s$ is *fair with respect to* E if the portion of points of $\mathcal{P}$ that belong to E is equal to the volume of E, i.e., if $A(E, N, \mathcal{P}) := \sum_{n=0}^{N-1} \chi_E(\boldsymbol{x}_n) = \lambda_s(E)N$. We say that the point set $\mathcal{P}$ is fair with respect to a nonempty collection $\mathcal{E}$ of Borel sets in $[0,1]^s$ if $\mathcal{P}$ is fair with respect to every $E \in \mathcal{E}$.

Let $\mathcal{E} = \{E_1, \dots, E_k\}$ be a partition of $[0,1]^s$ into nonempty Borel subsets of $[0,1]^s$. For a Lebesgue integrable function $f : [0,1]^s \to \mathbb{R}$ and for $1 \le j \le k$ put

$$G_j(f) := \sup_{\boldsymbol{t}\in E_j} f(\boldsymbol{t}) \quad \text{and} \quad g_j(f) := \inf_{\boldsymbol{t}\in E_j} f(\boldsymbol{t}).$$

Show that for any $\mathcal{P} = \{\boldsymbol{x}_0, \dots, \boldsymbol{x}_{N-1}\}$ which is fair with respect to $\mathcal{E}$ we have

$$\left| \int_{[0,1]^s} f(\boldsymbol{x}) \, \mathrm{d}x - \frac{1}{N} \sum_{n=0}^{N-1} f(\boldsymbol{x}_n) \right| \le \sum_{j=1}^{k} \lambda_s(E_j)(G_j(f) - g_j(f)).$$

Remark and Hint: This is a special case of [179, Theorem 2] where one can find a proof.

1.7 Let $f : [0,1]^s \to \mathbb{R}$ be continuous and let $\mathcal{E} = \{E_1, \dots, E_k\}$ be a partition of $[0,1]^s$ into nonempty Borel subsets of $[0,1]^s$. Show that for any $\mathcal{P} = \{\boldsymbol{x}_0, \dots, \boldsymbol{x}_{N-1}\}$ which is fair with respect to $\mathcal{E}$ we have

$$\left| \int_{[0,1]^s} f(\boldsymbol{x}) \, \mathrm{d}x - \frac{1}{N} \sum_{n=0}^{N-1} f(\boldsymbol{x}_n) \right| \le M_f(\delta(\mathcal{E})),$$

where $\delta(\mathcal{E}) := \max_{1\le j\le k} \sup_{\boldsymbol{x},\boldsymbol{y}\in E_j} |\boldsymbol{x} - \boldsymbol{y}|_\infty$. *Hint:* Compare with [179, Theorem 3].

1.8 Prove an analogue of Theorem 1.5 for functions $f : D \to \mathbb{R}$ defined on an integration domain $D \subset \mathbb{R}^s$ which has Lebesgue measure $0 < \lambda_s(D) < \infty$.

1.9 Let $f : [0,1]^2 \to \mathbb{R}$, $f(x_1, x_2) = 1$ if $x_1^2 + x_2^2 \leq 1$ and 0 otherwise. We are interested in $\int_0^1 \int_0^1 f(x_1, x_2)\, \mathrm{d}x_1\, \mathrm{d}x_2$ (which is $\pi/4$). Write a computer program (for instance with MATHEMATICA) which applies the MC method to this problem. Run some experiments and compare the integration error with $1/\sqrt{N}$, where N is the sample size.

1.10 Let $f : [0,1] \to \mathbb{R}$ and $g(x) = \frac{1}{2}[f(x) + f(1-x)]$. Show that $\sigma^2(g) \leq \frac{1}{2}\sigma^2(f)$. *Hint:* This is [175, Proposition 1.3].

2

Quasi-Monte Carlo integration, discrepancy and reproducing kernel Hilbert spaces

In this chapter we motivate the ideas behind concepts such as discrepancy, uniform distribution, quasi-Monte Carlo algorithms and others from the point of view of numerical integration. Most discrepancies considered here can be derived from numerical integration and can therefore be understood as worst-case errors of numerical integration of functions from certain function spaces. Using reproducing kernel Hilbert spaces as function spaces removes many technicalities and gives a nice pathway to the connections between discrepancies and worst-case errors of numerical integration.

2.1 Quasi-Monte Carlo rules

We consider the problem of integrating a high dimensional Lebesgue integrable function $f : [0,1]^s \to \mathbb{R}$ where this cannot be done analytically and therefore one has to resort to numerical algorithms. Indeed we consider the simplest of possible algorithms, namely we approximate

$$\int_{[0,1]^s} f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \approx \frac{1}{N} \sum_{n=0}^{N-1} f(\boldsymbol{x}_n), \tag{2.1}$$

where $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1} \in [0,1]^s$ are the quadrature points which one needs to choose. Because the volume of the unit cube $[0,1]^s$ is one, the value of the integral is just the average value of the function, which is exactly what the algorithm tries to approximate.

If the quadrature points $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1} \in [0,1]^s$ are chosen deterministically, the algorithm $\frac{1}{N} \sum_{n=0}^{N-1} f(\boldsymbol{x}_n)$ is called a *quasi-Monte Carlo (QMC) algorithm* or a *QMC rule*. On the surface the algorithm looks simple, but of course, the difficulty is how to choose the quadrature points. The following two main questions arise from this: how can we assess the quality of

some given quadrature points? And, how can we find quadrature points of particularly high quality?

In order to answer these questions, we need to specify which integrands $f : [0,1]^s \to \mathbb{R}$ we want to consider. Indeed, we want our algorithm to work not just for one specific integrand, but for a whole class of functions, that is, for a set of functions which have certain properties, so that, if we know that the integrand satisfies a certain property, i.e. is "smooth", then we know that the method we use works well. In other words, the point set is chosen a priori and we apply the QMC algorithm to an arbitrary function belonging to a certain class.

As we know from classical integration rules in dimension $s = 1$, like Simpson's Rule, the smoother the integrand the faster the error (which for QMC rules is given by $|\int_{[0,1]^s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \frac{1}{N}\sum_{n=0}^{N-1} f(\boldsymbol{x}_n)|$) goes to zero as N increases. The same can of course be observed for QMC rules. We first develop the classical theory on QMC methods, which deals with integrands of bounded variation [175]. In order to avoid too many technicalities though, we deal with absolutely continuous functions with partial first derivatives which are square integrable, instead of functions of bounded variation (see [175, p. 19] for an equivalence or [37, Section 3.1] for a discussion of the similarities between those two concepts).

2.2 Numerical integration in one dimension

As a first example, consider a one-dimensional function $f : [0,1] \to \mathbb{R}$ with continuous first derivative which is bounded on $[0,1]$. For a subset $J \subseteq [0,1]$ let $\chi_J(x)$ denote the characteristic function of J, i.e.,

$$\chi_J(x) = \begin{cases} 1 & \text{if } x \in J, \\ 0 & \text{if } x \notin J. \end{cases}$$

Considering the integration error of a QMC rule using a point set $\mathcal{P} = \{x_0, \ldots, x_{N-1}\} \subseteq [0,1]$, we obtain, by substituting $f(1) - \int_x^1 f'(y)\,\mathrm{d}y$ for $f(x)$, that

$$\begin{aligned} &\int_0^1 f(x)\,\mathrm{d}x - \frac{1}{N}\sum_{n=0}^{N-1} f(x_n) \\ &= \frac{1}{N}\sum_{n=0}^{N-1}\int_{x_n}^1 f'(y)\,\mathrm{d}y - \int_0^1\int_x^1 f'(y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_0^1 \frac{1}{N}\sum_{n=0}^{N-1} \chi_{(x_n,1]}(y) f'(y)\,\mathrm{d}y - \int_0^1\int_0^y f'(y)\,\mathrm{d}x\,\mathrm{d}y \end{aligned}$$

$$= \int_0^1 f'(y) \left[\frac{1}{N} \sum_{n=0}^{N-1} \chi_{(x_n,1]}(y) - y \right] \mathrm{d}y.$$

(Note that alternatively we could use $\chi_{[x_n,1]}(y)$ instead of $\chi_{(x_n,1]}(y)$, but the latter is used more commonly.) Note that

$$\sum_{n=0}^{N-1} \chi_{(x_n,1]}(y) = \sum_{n=0}^{N-1} \chi_{[0,y)}(x_n) =: A([0,y), N, \mathcal{P}),$$

the number of points of $\mathcal{P}$ which lie in the interval $[0,y)$. The expression between the squared brackets above leads to the following definition.

Definition 2.1 For a point set $\mathcal{P}$ consisting of N points in $[0,1)$ the function $\Delta_{\mathcal{P}} : [0,1] \to \mathbb{R}$,

$$\Delta_{\mathcal{P}}(y) := \frac{A([0,y), N, \mathcal{P})}{N} - y$$

is called the *discrepancy function* of $\mathcal{P}$.

The discrepancy function permits a geometric interpretation which gives us some insight. Namely: $A([0,y), N, \mathcal{P})/N$ is the proportion of points of $\mathcal{P}$ which lie in the interval $[0,y)$. The length or Lebesgue measure of the interval $[0,y)$ is of course y and so, for a given $y \in [0,1]$, the function $\Delta_{\mathcal{P}}(y)$ measures the difference between the proportion of points of $\mathcal{P}$ in the interval $[0,y)$ and the length of the interval $[0,y)$. We see that the discrepancy function is small when the points $x_0, \ldots, x_{N-1}$ are evenly spread over the interval $[0,1]$. A more detailed discussion of this geometric interpretation is given in Section 3.

Hence we have

$$\int_0^1 f(x)\,\mathrm{d}x - \frac{1}{N} \sum_{n=0}^{N-1} f(x_n) = \int_0^1 f'(y) \Delta_{\mathcal{P}}(y)\,\mathrm{d}y. \tag{2.2}$$

This equation is a simplified form of Hlawka's identity [110], which is also known as Zaremba's identity [270].

Thus the criterion for $\mathcal{P}$ should be to choose it such that $\Delta_{\mathcal{P}}(y)$ is small for all $y \in [0,1]$, then (2.2) guarantees that the error committed by $\mathcal{P}$ is also small for the class of functions which have continuous first derivative. To make the statement "$\Delta_{\mathcal{P}}(y)$ small for all $y \in [0,1]$" more tangible, we can take the absolute value on both sides of (2.2) and apply Hölder's inequality

to the right hand side to obtain

$$\left|\int_0^1 f(x)\,\mathrm{d}x - \frac{1}{N}\sum_{n=0}^{N-1} f(x_n)\right| \tag{2.3}$$
$$\leq \int_0^1 |f'(y)||\Delta_{\mathcal{P}}(y)|\,\mathrm{d}y \leq \left(\int_0^1 |f'(y)|^q\,\mathrm{d}y\right)^{1/q}\left(\int_0^1 |\Delta_{\mathcal{P}}(y)|^p\,\mathrm{d}y\right)^{1/p}$$

for $p, q \geq 1$ and $1/p + 1/q = 1$.

The last inequality (2.3) now separates the effects of the function and the point set on the integration error. Note that $\left(\int_0^1 |f'(y)|^q\,\mathrm{d}y\right)^{1/q}$ is a semi-norm on the function space, while $\|f\|_q := \left(|f(1)|^q + \int_0^1 |f'(y)|^q\,\mathrm{d}y\right)^{1/q}$ is a norm on the function space.

Two choices of p received particular attention, namely, $p = \infty$ and $p = 2$.

Definition 2.2 Let $\mathcal{P} = \{x_0, \ldots, x_{N-1}\}$ be a point set in the unit-interval $[0, 1)$. The *star discrepancy* of $\mathcal{P}$ is defined as

$$D_N^*(\mathcal{P}) := \sup_{y\in[0,1]} |\Delta_{\mathcal{P}}(y)|$$

and the L_2-*discrepancy* of $\mathcal{P}$ is defined as

$$L_{2,N}(\mathcal{P}) := \left(\int_0^1 |\Delta_{\mathcal{P}}(y)|^2\,\mathrm{d}y\right)^{1/2}.$$

From the definition of the discrepancy function, we can now see that the star discrepancy $D_N^*(\mathcal{P})$ and the L_2-discrepancy $L_{2,N}(\mathcal{P})$ of a point set $\mathcal{P}$ are small if the points in $\mathcal{P}$ are evenly spread over the interval $[0, 1]$ (see Exercise 2.1).

We can write (2.3) as

$$\left|\int_0^1 f(x)\,\mathrm{d}x - \frac{1}{N}\sum_{n=0}^{N-1} f(x_n)\right| \leq \|f\|_1 D_N^*(\mathcal{P}) \tag{2.4}$$

for $p = \infty$ and $q = 1$ and for $p = q = 2$ we can write

$$\left|\int_0^1 f(x)\,\mathrm{d}x - \frac{1}{N}\sum_{n=0}^{N-1} f(x_n)\right| \leq \|f\|_2 L_{2,N}(\mathcal{P}). \tag{2.5}$$

We remark that (2.4) is a simplified version of Koksma's inequality (see [128, Theorem 5.1] for the original version).

Remark 2.3 Note that (2.4) and (2.5) are slightly weaker than (2.3) because we switched from a semi-norm to a norm. On the other hand, the QMC algorithm integrates all constant functions exactly so that for all $c \in \mathbb{R}$ we have

$$\begin{aligned}\left|\int_0^1 f(x)\,\mathrm{d}x - \frac{1}{N}\sum_{n=0}^{N-1} f(x_n)\right| &= \left|\int_0^1 (f(x)-c)\,\mathrm{d}x - \frac{1}{N}\sum_{n=0}^{N-1}(f(x_n)-c)\right| \\ &\le \left(|f(1)-c|^q + \int_0^1 |f'(y)|^q\,\mathrm{d}y\right)^{1/q} \\ &\quad \times \left(\int_0^1 |\Delta_{\mathcal{P}}(y)|^p\,\mathrm{d}y\right)^{1/p}.\end{aligned}$$

By choosing $c = f(1)$ we see from the last expression that in our error analysis we could restrict ourselves to consider only functions for which $f(1) = 0$.

We now aim to develop this theory for arbitrary dimensions $s \ge 1$. Reproducing kernel Hilbert spaces make this generalisation somewhat simpler, hence we introduce them in the following section.

2.3 Reproducing kernel Hilbert spaces

Before we introduce reproducing kernel Hilbert spaces in a general setting, we work out an example which we already used implicitly in the previous section.

A first example

As we have seen from the one-dimensional example, the error analysis hinges on the substitution

$$f(x) = f(1) - \int_x^1 f'(y)\,\mathrm{d}y, \tag{2.6}$$

i.e., the analysis works for all functions which have such an integral representation. For functions f, g permitting such a substitution, and for which $f', g' \in L_2([0,1])$, we can introduce an inner product by using the value of f, g at one and the derivatives of f, g, that is

$$\langle f, g\rangle := f(1)g(1) + \int_0^1 f'(x)g'(x)\,\mathrm{d}x. \tag{2.7}$$

The corresponding norm $\|f\|_2 := \sqrt{\langle f, f\rangle}$ is exactly the norm used in (2.5). This defines a Hilbert space

$$\mathscr{H} = \{f : [0,1] \to \mathbb{R} : f \text{ absolutely continuous and } \|f\|_2 < \infty\}$$

(hence the fundamental theorem of calculus applies) whose first derivative is square integrable. With this, we can introduce a proper criterion for how well a given QMC rule $Q_N(f) = \frac{1}{N}\sum_{n=0}^{N-1} f(x_n)$ works by looking at the worst performance of Q_N for all functions in $\mathscr{H}$ with norm at most one.

Definition 2.4 Let $\mathscr{H}$ be a Hilbert space of Lebesgue integrable functions on $[0,1]$ with norm $\|\cdot\|$ for which function values are well-defined, and let $\mathcal{P}$ be the quadrature points used in the QMC rule Q_N. The *worst-case error* for QMC integration in the Hilbert space $\mathscr{H}$ is then given by

$$e(\mathscr{H}, \mathcal{P}) = \sup_{f\in\mathscr{H}, \|f\|\le 1} \left| \int_0^1 f(x)\,\mathrm{d}x - Q_N(f) \right|.$$

A particularly nice theory now develops when we combine Equation (2.6) and (2.7), i.e., for each $y \in [0,1]$ we want to have a function $g_y : [0,1] \to \mathbb{R}$ such that $\langle f, g_y\rangle = f(y)$. As we modelled the inner product after (2.6) in the first place (we used $f(1)$ and f' which both appear in (2.6)), it is not hard to see that this can be done. Indeed, $g_y(1) = 1$ for all $y \in [0,1]$ and $g_y'(x) = \frac{\mathrm{d}g_y}{\mathrm{d}x} = -1$ for all $x \in [y,1]$ and $g_y'(x) = 0$ for $x \in [0,y)$. This implies that g_y has to be of the form

$$g_y(x) = 2 - \begin{cases} c & \text{for } 0 \le x < y, \\ x & \text{for } y \le x \le 1, \end{cases}$$

for some arbitrary fixed constant $c \in \mathbb{R}$.

We add one more sensible condition on g_y, namely, that $g_y \in \mathscr{H}$ for each $y \in [0,1]$. Then the condition that g_y is an absolutely continuous function of x completely determines g_y, and we obtain that $c = y$, i.e.

$$g_y(x) = 2 - \begin{cases} y & \text{for } 0 \le x < y, \\ x & \text{for } y \le x \le 1, \end{cases}$$

which we can write as $g_y(x) = 2 - \max(x,y) = 1 + \min(1-x, 1-y)$.

To summarise, for each $y \in [0,1]$ we now have a function $g_y \in \mathscr{H}$ such that $\langle f, g_y\rangle = f(y)$. The function $(x,y) \mapsto g_y(x)$ is called a *reproducing kernel* [4] and has several useful properties. In the following we denote the reproducing kernel by K, so in our case

$$K(x,y) = g_y(x) = 1 + \min(1-x, 1-y).$$

Definition 2.5 A Hilbert space $\mathscr{H}$ of functions $f : X \to \mathbb{R}$ on a set X with inner product $\langle \cdot, \cdot \rangle$ is called a *reproducing kernel Hilbert space*, if there exists a function $K : X \times X \to \mathbb{R}$ such that

P1: $K(\cdot, y) \in \mathscr{H}$ for each fixed $y \in X$ and
P2: $\langle f, K(\cdot, y) \rangle = f(y)$ for each fixed $y \in X$ and for all $f \in \mathscr{H}$.

Note that here we consider K as a function of the first variable denoted by $\cdot$ and in $\langle f, K(\cdot, y) \rangle$ the inner product is taken with respect to the first variable of K. Sometimes we indicate this by writing $\langle f(x), K(x, y) \rangle_x$. The last property, i.e. P2, is the *reproducing property*, i.e. the function values of f can be reproduced via the kernel and the inner product.

It follows that a function K with these properties must also be symmetric, unique and positive semidefinite:

P3 (symmetry): this holds as

$$K(x, y) = \langle K(\cdot, y), K(\cdot, x) \rangle = \langle K(\cdot, x), K(\cdot, y) \rangle = K(y, x),$$

P4 (uniqueness): this holds since for any function $\widetilde{K}$ satisfying P1 and P2 we have

$$\widetilde{K}(x, y) = \langle \widetilde{K}(\cdot, y), K(\cdot, x) \rangle = \langle K(\cdot, x), \widetilde{K}(\cdot, y) \rangle = K(y, x) = K(x, y),$$

P5 (positive semidefiniteness): this holds as for all choices of $a_0, \ldots, a_{N-1} \in \mathbb{R}$ and $x_0, \ldots, x_{N-1} \in X$ we have

$$\begin{aligned} \sum_{m,n=0}^{N-1} a_m a_n K(x_m, x_n) &= \sum_{m,n=0}^{N-1} a_m a_n \langle K(\cdot, x_n), K(\cdot, x_m) \rangle \\ &= \left\langle \sum_{n=0}^{N-1} a_n K(x_n, \cdot), \sum_{m=0}^{N-1} a_m K(x_m, \cdot) \right\rangle \\ &= \left\| \sum_{m=0}^{N-1} a_m K(x_m, \cdot) \right\|^2 \geq 0. \end{aligned}$$

As was shown in [4], a function K which satisfies P3 and P5 also uniquely determines a Hilbert space of functions together with an inner product for which P1 and P2 (and hence also P4) hold. Thus it makes sense to speak of a reproducing kernel without explicitly specifying a Hilbert space of functions.

Remark 2.6 In our example, according to the construction of $K(x, y) = g_y(x)$ the conditions P1 and P2 are satisfied and hence $\mathscr{H} = \{f : [0, 1] \to$

$\mathbb{R} : \|f\|_2 < \infty\}$ is a reproducing kernel Hilbert space. We wrote $K(x,y) = 1 + \min(1-x, 1-y)$ rather than $K(x,y) = 2 - \max(x,y)$, as the function $\min(1-x, 1-y)$ is a reproducing kernel of the Hilbert space of absolutely continuous functions with square integrable first derivative for which $f(1) = 0$ for all f in this space (see Exercise 2.5).

Remark 2.7 We note that if we include complex functions $f : X \to \mathbb{C}$, then $\langle f, g\rangle = \overline{\langle g, f\rangle}$, $\langle f, ag\rangle = \overline{a}\langle f, g\rangle$ for $a \in \mathbb{C}$, P3 becomes $K(x,y) = \overline{K(y,x)}$ and we call a function positive semidefinite if for all choices of $a_0, \ldots, a_{N-1} \in \mathbb{C}$ and $x_0, \ldots, x_{N-1} \in X$ we have

$$\sum_{m,n=0}^{N-1} \overline{a}_m a_n K(x_m, x_n) \geq 0.$$

Example 2.8 We give another example of a reproducing kernel Hilbert space which was considered in [50]. This reproducing kernel Hilbert space is based on Walsh functions.

We recall some notation from Appendix A. Assume that $x, y \in [0,1)$ have b-adic expansion $x = \xi_1 b^{-1} + \xi_2 b^{-2} + \cdots$ and $y = \eta_1 b^{-1} + \eta_2 b^{-2} + \cdots$. Further let $k \in \mathbb{N}_0$ have b-adic expansion $k = \kappa_0 + \kappa_1 b + \cdots + \kappa_{a-1} b^{a-1}$. Further let $\omega_b = \mathrm{e}^{2\pi \mathrm{i}/b}$. Then the kth Walsh function in base b is defined by

$${}_b\mathrm{wal}_k(x) = \omega_b^{\kappa_0\xi_1 + \kappa_1\xi_2 + \cdots + \kappa_{a-1}\xi_a}.$$

Further we set

$$x \ominus y = \frac{\zeta_1}{b} + \frac{\zeta_2}{b^2} + \cdots,$$

where $\zeta_j = \xi_j + \eta_j \pmod b$ for all $j \geq 0$. See Appendix A for more information on Walsh functions.

Let $K_{\mathrm{wal}}(x,y) = \sum_{k=0}^{\infty} r_{\mathrm{wal},b,\alpha}(k)\, {}_b\mathrm{wal}_k(x \ominus y)$, where $r_{\mathrm{wal},b,\alpha}(0) = 1$ and for $k > 0$ with base b ($b \geq 2$) representation $k = \kappa_0 + \kappa_1 b + \cdots + \kappa_{a-1} b^{a-1}$ and $\kappa_{a-1} \neq 0$, we define $r_{\mathrm{wal},b,\alpha}(k) = b^{-\alpha a}$, where $\alpha > 1$. The reproducing kernel Hilbert space with kernel K_{wal} is called a *Walsh space* and consists of Walsh series $\sum_{k=0}^{\infty} \widehat{f}(k)\, {}_b\mathrm{wal}_k(x)$. The inner product in this space for two Walsh series $f(x) = \sum_{k=0}^{\infty} \widehat{f}(k)\, {}_b\mathrm{wal}_k(x)$ and $g(x) = \sum_{k=0}^{\infty} \widehat{g}(k)\, {}_b\mathrm{wal}_k(x)$ is given by $\langle f, g\rangle = \sum_{k=0}^{\infty} r_{\mathrm{wal},b,\alpha}(k)^{-1} \widehat{f}(k)\overline{\widehat{g}(k)}$.

The reproducing property can be verified in the following way: the kth Walsh coefficient of $K_{\mathrm{wal}}(\cdot, y)$ (considered as a function of the first variable) is given by $r_{\mathrm{wal},b,\alpha}(k)\, {}_b\mathrm{wal}_k(\ominus y)$ and hence

$$\langle f, K_{\mathrm{wal}}(\cdot, y)\rangle = \sum_{k=0}^{\infty} \frac{\widehat{f}(k)\overline{r_{\mathrm{wal},b,\alpha}(k)}\,\overline{{}_b\mathrm{wal}_k(\ominus y)}}{r_{\mathrm{wal},b,\alpha}(k)} = \sum_{k=0}^{\infty} \widehat{f}(k)\, {}_b\mathrm{wal}_k(y) = f(y).$$

Numerical integration in one dimension revisited

Using the framework of reproducing kernel Hilbert spaces we can now revisit Section 2.2. Hence we define the reproducing kernel K as in Section 2.2 by

$$K(x,y) = 1 + \min(1-x, 1-y)$$

and the inner product by $\langle f, g\rangle = f(1)g(1) + \int_0^1 f'(x)g'(x)\,\mathrm{d}x$. We have

$$\int_0^1 f(y)\,\mathrm{d}y = \int_0^1 \langle f, K(\cdot,y)\rangle\,\mathrm{d}y = \left\langle f, \int_0^1 K(\cdot,y)\,\mathrm{d}y \right\rangle,$$

where the second equality is obtained by a change of the order of integration, and

$$Q_N(f) = \frac{1}{N}\sum_{n=0}^{N-1} f(x_n) = \frac{1}{N}\sum_{n=0}^{N-1} \langle f, K(\cdot,x_n)\rangle = \left\langle f, \frac{1}{N}\sum_{n=0}^{N-1} K(\cdot,x_n)\right\rangle,$$

where the inner product is taken with respect to the first variable of K. Thus, using the Cauchy-Schwarz Inequality, we have

$$\begin{aligned}\left|\int_0^1 f(y)\,\mathrm{d}y - Q_N(f)\right| &= \left|\left\langle f, \int_0^1 K(\cdot,y)\,\mathrm{d}y - \frac{1}{N}\sum_{n=0}^{N-1} K(\cdot,x_n)\right\rangle\right| \\ &\le \|f\|_2 \left\|\int_0^1 K(\cdot,y)\,\mathrm{d}y - \frac{1}{N}\sum_{n=0}^{N-1} K(\cdot,x_n)\right\|_2 . \quad (2.8)\end{aligned}$$

Note that we have $\Delta_{\mathcal{P}}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^1 K(x,y)\,\mathrm{d}y - \frac{1}{N}\sum_{n=0}^{N-1} K(x,x_n)\right)$ and hence

$$L_{2,N}(\mathcal{P}) = \left\|\int_0^1 K(\cdot,y)\,\mathrm{d}y - \frac{1}{N}\sum_{n=0}^{N-1} K(\cdot,x_n)\right\|_2 .$$

Let us now calculate the worst-case error. For short we write now $h(x) = \int_0^1 K(x,y)\,\mathrm{d}y - \frac{1}{N}\sum_{n=0}^{N-1} K(x,x_n)$. Since $K(\cdot,y) \in \mathscr{H}$ and also $\int_0^1 K(\cdot,y)\,\mathrm{d}y \in \mathscr{H}$ it is clear that $h \in \mathscr{H}$. We have equality in (2.8) if $f(x) = h(x)$. Let

$$\begin{aligned} e(f,\mathcal{P}) &:= \int_0^1 f(y)\,\mathrm{d}y - Q_N(f) \\ &= \left\langle f, \int_0^1 K(\cdot,y)\,\mathrm{d}y - \frac{1}{N}\sum_{n=0}^{N-1} K(\cdot,x_n)\right\rangle = \langle f, h\rangle. \end{aligned}$$

Then for all f with $\|f\|_2 \neq 0$ we have $\frac{e(f,\mathcal{P})}{\|f\|_2} = e(f/\|f\|_2, \mathcal{P}) \le e(h/\|h\|_2, \mathcal{P}) =$

$\frac{e(h,\mathcal{P})}{\|h\|_2}$ by a property of the inner product and hence

$$e(\mathscr{H},\mathcal{P}) = \frac{e(h,\mathcal{P})}{\|h\|_2} = \frac{\langle h,h\rangle}{\|h\|_2} = \|h\|_2.$$

This means, that for a given point set $\mathcal{P}$, among all functions in the space $\mathscr{H}$, the function $h \in \mathscr{H}$ is the hardest to integrate. For the function h we have equality in (2.8).

The worst-case error for arbitrary reproducing kernel Hilbert spaces

In the following we use the approach of Hickernell [99] and Sloan & Woźniakowski [247]. Let us now consider an arbitrary Hilbert space $\mathscr{H}$ of Lebesgue integrable functions $f : [0,1]^s \to \mathbb{R}$, $s \geq 1$, with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$. Consider the functional $T_{\boldsymbol{y}}$ which evaluates a function at the point $\boldsymbol{y}$, i.e.

$$T_{\boldsymbol{y}}(f) = f(\boldsymbol{y}) \quad \forall f \in \mathscr{H}.$$

Because we want to approximate the integral $\int_{[0,1]^s} f(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}$ by the average of some function values $\frac{1}{N}\sum_{n=0}^{N-1} f(\boldsymbol{x}_n)$, it is reasonable to demand that $|f(\boldsymbol{x}_n)| < \infty$, which is ensured by the condition that the functional $T_{\boldsymbol{y}}$ is bounded, i.e., that there is an $M < \infty$ such that $|T_{\boldsymbol{y}}(f)| \leq M$ for all $f \in \mathscr{H}$ with $\|f\| \leq 1$. Riesz' representation theorem now implies that there exists a unique function $K(\cdot,\boldsymbol{y}) \in \mathscr{H}$ such that $T_{\boldsymbol{y}}(f) = \langle f, K(\cdot,\boldsymbol{y})\rangle$ for all $f \in \mathscr{H}$. Properties P1 and P2 now imply that K is the reproducing kernel for the Hilbert space $\mathscr{H}$ (and hence $\mathscr{H}$ is a reproducing kernel Hilbert space).

An essential property which we used in the previous section is the fact that

$$\int_0^1 \langle f, K(\cdot,y)\rangle\,\mathrm{d}y = \left\langle f, \int_0^1 K(\cdot,y)\,\mathrm{d}y \right\rangle$$

for the reproducing kernel $K(x,y) = 1 + \min(1-x, 1-y)$, as this represents only a change of the order of integration. As changing the order of integration and inner product is essential for our error analysis, we consider in the following under which conditions this holds for arbitrary reproducing kernels.

Let now T be another bounded linear functional on $\mathscr{H}$ (not necessarily integration), then, again by the Riesz representation theorem, it follows that there exists a unique function $R \in \mathscr{H}$ such that $T(f) = \langle f, R\rangle$ for all $f \in \mathscr{H}$.

On the other hand we have

$$R(\boldsymbol{x}) = \langle R, K(\cdot, \boldsymbol{x})\rangle = \langle K(\cdot, \boldsymbol{x}), R\rangle = T(K(\cdot, \boldsymbol{x})),$$

where in the first equality we used the reproducing property of K and in the third equality we used that R is the representer of the functional T (note that $R \in \mathscr{H}$ and for any given $\boldsymbol{x}$ also $K(\cdot, \boldsymbol{x}) \in \mathscr{H}$). Here the inner product and the operator T are applied to the first variable of K. Thus, for any bounded linear functional T we have

$$T(\langle f(\boldsymbol{x}), K(\boldsymbol{x}, \boldsymbol{y})\rangle_{\boldsymbol{x}}) = T(f) = \langle f, R\rangle = \langle f(\boldsymbol{x}), T(K(\boldsymbol{y}, \boldsymbol{x}))\rangle_{\boldsymbol{x}},$$

where the inner product is always with respect to the variable $\boldsymbol{x}$ (which is indicated by writing $\langle\cdot,\cdot\rangle_{\boldsymbol{x}}$ instead of $\langle\cdot,\cdot\rangle$) and the operator T is always applied to the variable $\boldsymbol{y}$.

Example 2.9 Consider the operator $I(f) = \int_{[0,1]^s} f(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}$. First we have

$$I(f) = \int_{[0,1]^s} f(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = \int_{[0,1]^s} \langle f, K(\cdot, \boldsymbol{y})\rangle\,\mathrm{d}\boldsymbol{y}.$$

By the above, the representer R of the functional I is given by

$$R(\boldsymbol{x}) = I(K(\cdot, \boldsymbol{x})) = \int_{[0,1]^s} K(\boldsymbol{y}, \boldsymbol{x})\,\mathrm{d}\boldsymbol{y} = \int_{[0,1]^s} K(\boldsymbol{x}, \boldsymbol{y})\,\mathrm{d}\boldsymbol{y}.$$

Hence we obtain

$$\int_{[0,1]^s} \langle f, K(\cdot, \boldsymbol{y})\rangle\,\mathrm{d}\boldsymbol{y} = I(f) = \langle f, R\rangle = \left\langle f, \int_{[0,1]^s} K(\cdot, \boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\right\rangle. \tag{2.9}$$

Hence integral and inner product in a reproducing kernel Hilbert space can always be interchanged as long as the integration functional I is bounded.

We are especially interested in two operators:

- the integration operator $I(f) := \int_{[0,1]^s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$ and
- the QMC rule $Q_N(f) := \frac{1}{N}\sum_{n=0}^{N-1} f(\boldsymbol{x}_n)$ using the quadrature points $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1} \in [0,1]^s$.

For an arbitrary f with $\|f\| \neq 0$ we have

$$\begin{aligned}\frac{|f(\boldsymbol{y})|}{\|f\|} &= |\langle f/\|f\|, K(\cdot, \boldsymbol{y})\rangle| \\ &\le \langle K(\cdot, \boldsymbol{y})/\|K(\cdot, \boldsymbol{y})\|, K(\cdot, \boldsymbol{y})\rangle \\ &= \sqrt{\langle K(\cdot, \boldsymbol{y}), K(\cdot, \boldsymbol{y})\rangle} = \sqrt{K(\boldsymbol{y}, \boldsymbol{y})}.\end{aligned}$$

Thus we have $|T_{\boldsymbol{y}}(f)|/\|f\| \leq \sqrt{K(\boldsymbol{y},\boldsymbol{y})}$ and that

$$|I(f)|/\|f\| \leq \int_{[0,1]^s} |f(\boldsymbol{y})| \,\mathrm{d}\boldsymbol{y}/\|f\| \leq \int_{[0,1]^s} \sqrt{K(\boldsymbol{y},\boldsymbol{y})} \,\mathrm{d}\boldsymbol{y}$$

for all $f \in \mathscr{H}$ with $\|f\| \neq 0$.

First note that reproducing kernel Hilbert spaces are defined as Hilbert spaces of functions in which pointwise evaluation is a continuous linear functional, in other words, in which point evaluation is a bounded linear functional as introduced at the beginning of this subsection. As $K(\cdot,\boldsymbol{y}) \in \mathscr{H}$ we have $K(\boldsymbol{y},\boldsymbol{y}) < \infty$ for all $\boldsymbol{y} \in [0,1]^s$ by the definition of reproducing kernel Hilbert spaces. Hence $|f(\boldsymbol{y})| \leq \|f\|\sqrt{K(\boldsymbol{y},\boldsymbol{y})} < \infty$ and the QMC rule is well defined for integrands which lie in some reproducing kernel Hilbert space.

If a reproducing kernel also satisfies

C: $\int_{[0,1]^s} \sqrt{K(\boldsymbol{y},\boldsymbol{y})} \,\mathrm{d}\boldsymbol{y} < \infty$,

then, by the above, the integration operator and the QMC rule are both bounded linear functionals. In this case (2.9) always holds.

Like the reproducing kernel from the previous section, the other reproducing kernels considered in this book also satisfy condition C.

Definition 2.10 Let $\mathscr{H}$ be a reproducing kernel Hilbert space for which I is a bounded linear functional. Then the *initial error* is defined as

$$e(\mathscr{H},0) = \|I\| = \sup_{f\in\mathscr{H},\|f\|\leq 1} |I(f)|$$

and the *worst-case error* for a QMC rule based on the quadrature points $\mathcal{P} = \{\boldsymbol{x}_0,\dots,\boldsymbol{x}_{N-1}\} \subseteq [0,1]^s$ is defined as

$$e(\mathscr{H},\mathcal{P}) = \|I - Q_N\| = \sup_{f\in\mathscr{H},\|f\|\leq 1} |I(f) - Q_N(f)|.$$

The initial error is introduced as a reference. We always assume that the initial error is finite, which is equivalent to saying that the integral operator is bounded.

With this, the same error analysis as in the previous section applies, namely:

$$I(f) = \left\langle f, \int_{[0,1]^s} K(\cdot,\boldsymbol{y}) \,\mathrm{d}\boldsymbol{y} \right\rangle,$$

where we used the fact that the representer for the functional I is given by

$I(K(\cdot,\boldsymbol{x})) = \int_{[0,1]^s} K(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}$ and

$$Q_N(f) = \left\langle f, \frac{1}{N}\sum_{n=0}^{N-1} K(\cdot,\boldsymbol{x}_n)\right\rangle,$$

where we used that the representer for the functional Q_N is given by $Q_N(K(\cdot,\boldsymbol{x})) = \frac{1}{N}\sum_{n=0}^{N-1} K(\boldsymbol{x},\boldsymbol{x}_n)$.

The initial error is thus given by

$$\begin{aligned} e(\mathscr{H},0) = \|I\| &= \sup_{f\in\mathscr{H},\|f\|\le 1} |I(f)| \\ &= \sup_{f\in\mathscr{H},\|f\|\le 1} \left|\left\langle f, \int_{[0,1]^s} K(\cdot,\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\right\rangle\right| \\ &= \sqrt{\left\langle \int_{[0,1]^s} K(\cdot,\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}, \int_{[0,1]^s} K(\cdot,\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\right\rangle}, \end{aligned}$$

since the largest value of the supremum occurs for $\frac{g}{\|g\|}$, where $g(\boldsymbol{x}) = \int_{[0,1]^s} K(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} \in \mathscr{H}$ is the representer of the integration functional. Therefore we have

$$\begin{aligned} e^2(\mathscr{H},0) = \|I\|^2 &= \int_{[0,1]^s}\int_{[0,1]^s} \langle K(\cdot,\boldsymbol{x}), K(\cdot,\boldsymbol{y})\rangle\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} \\ &= \int_{[0,1]^{2s}} K(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}. \end{aligned}$$

The integration error is given by

$$I(f) - Q_N(f) = \langle f, h\rangle, \tag{2.10}$$

where the *representer* of the integration error is given by

$$h(\boldsymbol{x}) = \int_{[0,1]^s} K(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} - \frac{1}{N}\sum_{n=0}^{N-1} K(\boldsymbol{x},\boldsymbol{x}_n).$$

We can estimate this error using the Cauchy-Schwarz Inequality with

$$|I(f) - Q_N(f)| \le \|f\|\|h\|.$$

From (2.10) it is then clear that the function in the unit ball of $\mathscr{H}$ which is hardest to integrate is $h/\|h\|$ and hence the worst-case error is given by

$$e(\mathscr{H},\mathcal{P}) = \|h\|.$$

For the square worst-case error $e^2(\mathscr{H},\mathcal{P}) = \langle h, h\rangle$.

Proposition 2.11 *Let $\mathscr{H}$ be a reproducing kernel Hilbert space whose reproducing kernel K satisfies condition* C. *Then the square initial error is given by*

$$e^2(\mathscr{H},\mathcal{P}) = \int_{[0,1]^{2s}} K(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}$$

and the square worst-case error for QMC integration of functions from $\mathscr{H}$ using the quadrature points $\mathcal{P} = \{\boldsymbol{x}_0,\dots,\boldsymbol{x}_{N-1}\}$ is given by

$$e^2(\mathscr{H},\mathcal{P}) = \int_{[0,1]^{2s}} K(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} - \frac{2}{N}\sum_{n=0}^{N-1}\int_{[0,1]^s} K(\boldsymbol{x}_n,\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} + \frac{1}{N^2}\sum_{n,m=0}^{N-1} K(\boldsymbol{x}_n,\boldsymbol{x}_m).$$

We give a modification of a classical result for the star discrepancy (see Proposition 3.16). The following result, called the *triangle inequality for the worst-case error*, which was first proved in [103], gives a bound for the worst-case error in $\mathscr{H}$ of a QMC rule using a point set $\mathcal{P}$ which is a superposition of several smaller point sets.

Lemma 2.12 *Let $\mathscr{H}$ be a reproducing kernel Hilbert space of functions on $[0,1]^s$. For $1 \le i \le k$ let $\mathcal{P}_i$ be point sets consisting of N_i points in $[0,1)^s$ with worst-case error $e(\mathscr{H},\mathcal{P}_i)$. Let $\mathcal{P}$ be the point set obtained by listing in some order the terms of $\mathcal{P}_i$, $1 \le i \le k$. We set $N = N_1 + \cdots + N_k$, which is the number of points of $\mathcal{P}$. Then we have*

$$e(\mathscr{H},\mathcal{P}) \le \sum_{i=1}^{k} \frac{N_i}{N} e(\mathscr{H},\mathcal{P}_i).$$

Proof We have

$$\begin{aligned} Ne(\mathscr{H},\mathcal{P}) &= \left\| N\int_{[0,1]^s} K(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} - \sum_{i=1}^{k}\sum_{\boldsymbol{y}\in\mathcal{P}_i} K(\boldsymbol{x},\boldsymbol{y}) \right\| \\ &\le \sum_{i=1}^{k} \left\| N_i\int_{[0,1]^s} K(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} - \sum_{\boldsymbol{y}\in\mathcal{P}_i} K(\boldsymbol{x},\boldsymbol{y}) \right\| \\ &= \sum_{i=1}^{k} N_i e(\mathscr{H},\mathcal{P}_i). \end{aligned}$$

□

The formulas in this section give us a convenient method for finding the worst-case and initial errors of arbitrary reproducing kernel Hilbert spaces.

In the following section we obtain some classical results by making use of reproducing kernel Hilbert spaces and the results in this section.

2.4 Connections to classical discrepancy theory

We now turn to classical results on numerical integration in arbitrary dimension $s \geq 1$, which we already considered for dimension $s = 1$ in Section 2.2. Let the quadrature points be given by $\mathcal{P} = \{\boldsymbol{x}_0, \dots, \boldsymbol{x}_{N-1}\}$, where $\boldsymbol{x}_n = (x_{n,1}, \dots, x_{n,s})$.

In the previous section we have already analysed the worst-case error for arbitrary reproducing kernel Hilbert spaces. An interesting as well as practical feature of the worst-case error is that we only need to know the reproducing kernel for the space to obtain formulas for the worst-case and initial error. To generalise from the one-dimensional case, considered at the beginning, to arbitrary high dimensions, we consider tensor product spaces of the one-dimensional spaces considered before. From [4, Section 8] we know that the reproducing kernel for this space is simply the product of the one-dimensional reproducing kernels. Hence, for the one-dimensional reproducing kernel $K(x, y) = \min(1 - x, 1 - y)$ considered in Section 2.3 (see in particular Remark 2.6), we obtain that the reproducing kernel of the s-fold tensor product is given by

$$K(\boldsymbol{x}, \boldsymbol{y}) = \prod_{i=1}^{s} K(x_i, y_i) = \prod_{i=1}^{s} \min(1 - x_j, 1 - y_j),$$

where $\boldsymbol{x} = (x_1, \dots, x_s), \boldsymbol{y} = (y_1, \dots, y_s) \in [0, 1]^s$.

What functions are in this space? The one-dimensional space contains all absolutely continuous functions $f : [0, 1] \to \mathbb{R}$ for which $f(1) = 0$ and the first derivative is square integrable. The inner product in one dimension is given by $\langle f, g \rangle = \int_0^1 f'(x) g'(x) \, \mathrm{d}x$.

For the tensor product space we then have, for example, if $f_1, \dots, f_s$ are functions in the one-dimensional space, then $f(x_1, \dots, x_s) = \prod_{i=1}^{s} f_i(x_i)$ is in the tensor-product space. The inner product of two such functions f and $g(x_1, \dots, x_s) = \prod_{i=1}^{s} g_i(x_i)$ is then

$$\langle f, g \rangle = \prod_{i=1}^{s} \langle f_i, g_i \rangle = \prod_{i=1}^{s} \int_0^1 f_i'(x_i) g_i'(x_i) \, \mathrm{d}x_i = \int_{[0,1]^s} \frac{\partial^s f}{\partial \boldsymbol{x}}(\boldsymbol{x}) \frac{\partial^s g}{\partial \boldsymbol{x}}(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}.$$

The tensor product space contains not only those products, and sums of those products, but also its completion with respect to the norm induced by

the inner product

$$\langle f, g\rangle = \int_{[0,1]^s} \frac{\partial^s f}{\partial \boldsymbol{x}}(\boldsymbol{x}) \frac{\partial^s g}{\partial \boldsymbol{x}}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.$$

Note that, as for the one-dimensional space we have $f(1) = 0$, it follows that $\frac{\partial^{|\mathfrak{u}|} f}{\partial \boldsymbol{x}_{\mathfrak{u}}}(\boldsymbol{x}_{\mathfrak{u}}, \mathbf{1}) = 0$ for all $\mathfrak{u} \subsetneq \mathcal{I}_s := \{1, \ldots, s\}$, where $(\boldsymbol{x}_{\mathfrak{u}}, \mathbf{1})$ is the vector whose ith component is x_i if $i \in \mathfrak{u}$ and 1 otherwise.

We now consider numerical integration in this space. From the previous section we know that

$$|I(f) - Q_N(f)| \le \|f\|\|h\|,$$

where $e(\mathscr{H}, \mathcal{P}) = \|h\|$ with

$$\begin{aligned} h(\boldsymbol{x}) &= \int_{[0,1]^s} K(\boldsymbol{x}, \boldsymbol{y})\,\mathrm{d}\boldsymbol{y} - \frac{1}{N}\sum_{n=0}^{N-1} K(\boldsymbol{x}, \boldsymbol{x}_n) \\ &= \prod_{i=1}^{s} \frac{1 - x_i^2}{2} - \frac{1}{N}\sum_{n=0}^{N-1}\prod_{i=1}^{s} \min(1 - x_i, 1 - x_{n,i}), \end{aligned}$$

where we used $\int_0^1 K(x_i, y_i)\,\mathrm{d}y_i = \int_0^1 \min(1 - x_i, 1 - y_i)\,\mathrm{d}y_i = (1 - x_i^2)/2$. Then

$$\frac{\partial^s}{\partial \boldsymbol{x}} h(\boldsymbol{x}) = (-1)^s \left(\prod_{i=1}^{s} x_i - \frac{1}{N}\sum_{n=0}^{N-1} \chi_{[\mathbf{0},\boldsymbol{x})}(\boldsymbol{x}_n) \right), \tag{2.11}$$

where $[\mathbf{0}, \boldsymbol{x})$ denotes the interval $\prod_{i=1}^{s}[0, x_i)$.

Apart from the factor $(-1)^s$, the right-hand side of (2.11) permits some geometrical interpretation. We write $A([\mathbf{0}, \boldsymbol{x}), N, \mathcal{P}) := \sum_{n=0}^{N-1} \chi_{[\mathbf{0},\boldsymbol{x})}(\boldsymbol{x}_n)$, which is the number of points of $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$ that belong to the interval $[\mathbf{0}, \boldsymbol{x})$.

Definition 2.13 For a point set $\mathcal{P}$ consisting of N points in $[0,1)^s$ the function $\Delta_{\mathcal{P}} : [0,1]^s \to \mathbb{R}$,

$$\Delta_{\mathcal{P}}(\boldsymbol{x}) = \frac{A([\mathbf{0}, \boldsymbol{x}), N, \mathcal{P})}{N} - \prod_{i=1}^{s} x_i$$

denotes the s-dimensional *discrepancy function* of $\mathcal{P}$.

It generalises the one-dimensional discrepancy function given in Definition 2.1 in Section 2.2. The geometrical interpretation also generalises from the one-dimensional example, i.e., it measures the difference between the proportion of points in a cube $[\mathbf{0}, \boldsymbol{x})$ and the volume of this cube.

Hence

$$e(\mathscr{H},\mathcal{P}) = \|h\| = \left(\int_{[0,1]^s} |\Delta_{\mathcal{P}}(\boldsymbol{x})|^2 \,\mathrm{d}\boldsymbol{x}\right)^{1/2},$$

and this is the classical L_2-discrepancy.

Definition 2.14 For a point set $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$ the *L_2-discrepancy* $L_{2,N}(\mathcal{P})$ is given by

$$L_{2,N}(\mathcal{P}) := \left(\int_{[0,1]^s} |\Delta_{\mathcal{P}}(\boldsymbol{x})|^2 \,\mathrm{d}\boldsymbol{x}\right)^{1/2},$$

and the *star discrepancy* is given by

$$D_N^*(\mathcal{P}) := \sup_{\boldsymbol{x}\in[0,1]^s} |\Delta_{\mathcal{P}}(\boldsymbol{x})|.$$

There is a concise formula for the classical L_2-discrepancy due to Warnock [261], which we derive in the following. We have $\int_0^1 K(x,y)\,\mathrm{d}y = \int_0^1 \min(1-x,1-y)\,\mathrm{d}y = (1-x^2)/2$ and $\int_0^1\int_0^1 K(x,y)\,\mathrm{d}x\,\mathrm{d}y = 1/3$. Thus Proposition 2.11 yields the following formula for the L_2-discrepancy.

Proposition 2.15 *For any point set $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$ in $[0,1]^s$ we have*

$$(L_{2,N}(\mathcal{P}))^2 = \frac{1}{3^s} - \frac{2}{N}\sum_{n=0}^{N-1}\prod_{i=1}^{s}\frac{1-x_{n,i}^2}{2} + \frac{1}{N^2}\sum_{m,n=0}^{N-1}\prod_{i=1}^{s}\min(1-x_{m,i},1-x_{n,i}),$$

where $x_{n,i}$ is the ith component of the point $\boldsymbol{x}_n$.

Remark 2.16 Using the formula in Proposition 2.15 the L_2-discrepancy of a point set consisting of N points in $[0,1)^s$ can be computed in $O(sN^2)$ operations. Based on this formula Heinrich [90] introduced an asymptotically even faster algorithm using $O(N(\log N)^s)$ operations for fixed s, which has been further improved to $O(N(\log N)^{s-1})$ operations by Frank & Heinrich [80]. It should be remarked that there is no concise formula which allows a computation of the star discrepancy (apart from the one-dimensional case, see [128, Chapter 2, Theorem 1.4] or [175, Theorem 2.6]). It was shown by Gnewuch, Srivastav & Winzen [85] that the computation of star discrepancy is an NP-hard problem. For a more detailed discussion of this topic we refer to [85] and the references therein.

The condition on the integrands is rather stringent. As we can see from the definition of the space, lower dimensional projections are ignored. Hence

one often considers the reproducing kernel Hilbert space with reproducing kernel

$$K(\boldsymbol{x},\boldsymbol{y}) = \prod_{i=1}^{s}(1+\min(1-x_i,1-y_i)).$$

Again from Section 2.2 we know that the inner product for the one-dimensional space is given by $\langle f,g\rangle = f(1)g(1)+\int_0^1 f'(x)g'(x)\,\mathrm{d}x$. Hence the inner product in the tensor product space $\mathcal{H}_s$ for functions $f(\boldsymbol{x}) = \prod_{i=1}^s f_i(x_i)$ and $g(\boldsymbol{x}) = \prod_{i=1}^s g_i(x_i)$ is then

$$\begin{aligned}\langle f,g\rangle &= \prod_{i=1}^{s}\langle f_i,g_i\rangle = \prod_{i=1}^{s}\left(f_i(1)g_i(1)+\int_0^1 f_i'(x_i)g_i'(x_i)\,\mathrm{d}x_i\right)\\ &= \sum_{\mathfrak{u}\subseteq\mathcal{I}_s}\int_{[0,1]^{|\mathfrak{u}|}}\prod_{i\in\mathfrak{u}}f_i'(x_i)\prod_{i\in\mathcal{I}_s\setminus\mathfrak{u}}f_i(1)\prod_{i\in\mathfrak{u}}g_i'(x_i)\prod_{i\in\mathcal{I}_s\setminus\mathfrak{u}}g_i(1)\,\mathrm{d}\boldsymbol{x}_{\mathfrak{u}}.\end{aligned}$$

In general the inner product for arbitrary functions f,g in this space is given by

$$\langle f,g\rangle = \sum_{\mathfrak{u}\subseteq\mathcal{I}_s}\int_{[0,1]^{|\mathfrak{u}|}}\frac{\partial^{|\mathfrak{u}|}f}{\partial\boldsymbol{x}_{\mathfrak{u}}}(\boldsymbol{x}_{\mathfrak{u}},\mathbf{1})\frac{\partial^{|\mathfrak{u}|}g}{\boldsymbol{x}_{\mathfrak{u}}}(\boldsymbol{x}_{\mathfrak{u}},\mathbf{1})\,\mathrm{d}\boldsymbol{x}_{\mathfrak{u}}.$$

From (2.10) we know that $I(f)-Q_N(f) = \langle f,h\rangle$, with

$$\begin{aligned}h(\boldsymbol{x}) &= \int_{[0,1]^s}K(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} - \frac{1}{N}\sum_{n=0}^{N-1}K(\boldsymbol{x},\boldsymbol{x}_n)\\ &= \prod_{i=1}^{s}\frac{3-x_i^2}{2} - \frac{1}{N}\sum_{n=0}^{N-1}\prod_{i=1}^{s}(1+\min(1-x_i,1-x_{n,i})).\end{aligned}$$

Then for $\mathfrak{u}\subseteq\mathcal{I}_s$ we have

$$\frac{\partial^{|\mathfrak{u}|}}{\partial\boldsymbol{x}_{\mathfrak{u}}}h(\boldsymbol{x}_{\mathfrak{u}},\mathbf{1}) = (-1)^{|\mathfrak{u}|}\left(\prod_{i\in\mathfrak{u}}x_i - \frac{1}{N}\sum_{n=0}^{N-1}\chi_{[\mathbf{0}_{\mathfrak{u}},\boldsymbol{x}_{\mathfrak{u}})}(\boldsymbol{x}_{n,\mathfrak{u}})\right),$$

where $[\mathbf{0}_{\mathfrak{u}},\boldsymbol{x}_{\mathfrak{u}})$ denotes the interval $\prod_{i\in\mathfrak{u}}[0,x_i)$. Note that $\frac{\partial^{|\mathfrak{u}|}}{\partial\boldsymbol{x}_{\mathfrak{u}}}h(\boldsymbol{x}_{\mathfrak{u}},\mathbf{1}) = (-1)^{|\mathfrak{u}|+1}\Delta_{\mathcal{P}}(\boldsymbol{x}_{\mathfrak{u}},\mathbf{1})$.

The following formula due to Hlawka [110] is called *Hlawka's identity* (but it is also known as *Zaremba's identity* [270]), and follows from $I(f)-Q_N(f) = \langle f,h\rangle$ by substitution.

Proposition 2.17 *The QMC integration error for any function $f\in\mathscr{H}_s$*

is given by

$$Q_N(f) - I(f) = \sum_{\mathfrak{u} \subseteq \mathcal{I}_s} (-1)^{|\mathfrak{u}|} \int_{[0,1]^{|\mathfrak{u}|}} \frac{\partial^{|\mathfrak{u}|} f}{\partial \boldsymbol{x}_{\mathfrak{u}}}(\boldsymbol{x}_{\mathfrak{u}}, \mathbf{1}) \Delta_{\mathcal{P}}(\boldsymbol{x}_{\mathfrak{u}}, \mathbf{1}) \, \mathrm{d}\boldsymbol{x}_{\mathfrak{u}}.$$

Note that we have $\Delta_{\mathcal{P}}(\boldsymbol{x}_{\emptyset}, \mathbf{1}) = \Delta_{\mathcal{P}}(\mathbf{1}) = 0$ and hence the case $\mathfrak{u} = \emptyset$ can be excluded in the above sum.

Applying the estimate $|\Delta_{\mathcal{P}}(\boldsymbol{x}_{\mathfrak{u}}, \mathbf{1})| \leq \sup_{\boldsymbol{x} \in [0,1]^s} |\Delta_{\mathcal{P}}(\boldsymbol{x})| = D_N^*(\mathcal{P})$, the star discrepancy of the point set $\mathcal{P}$, to Hlawka's identity we obtain the classical Koksma-Hlawka inequality.

Proposition 2.18 (Koksma-Hlawka inequality) *Let $\mathcal{P}$ be the quadrature points employed by the QMC rule Q_N and for a function $f : [0,1]^s \to \mathbb{R}$ for which all partial mixed derivatives are continuous on $[0,1]^s$ let $\|f\|_1 = \sum_{\mathfrak{u} \subseteq \mathcal{I}_s} \int_{[0,1]^{|\mathfrak{u}|}} \left| \frac{\partial^{|\mathfrak{u}|} f}{\partial \boldsymbol{x}_{\mathfrak{u}}}(\boldsymbol{x}_{\mathfrak{u}}, \mathbf{1}) \right| \mathrm{d}\boldsymbol{x}_{\mathfrak{u}}$. Then the integration error for functions with $\|f\|_1 < \infty$ can be bounded by*

$$|I(f) - Q_N(f)| \leq \|f\|_1 D_N^*(\mathcal{P}).$$

Remark 2.19 Koksma [120] proved the inequality for dimension $s = 1$ and Hlawka [109] generalised it to arbitrary dimension $s \geq 1$. Those inequalities in their original version consider functions of bounded variation in the sense of Hardy and Krause (which is, in the one-dimensional case, the same as the total variation) rather than functions f for which $\|f\|_1 < \infty$. The variation in the sense of Hardy and Krause and the norm considered here, without the summand $|f(\mathbf{1})|$, coincide whenever all the mixed partial derivatives are continuous on $[0,1]^s$, see for example [175, p. 19] or [37, Section 3.1].

Further information concerning the relationship between integration and discrepancy can be found in the books of Novak & Woźniakowski [198, 200] and of Triebel [256].

2.5 Numerical integration in weighted spaces

We now generalise the function spaces considered above based on ideas from Sloan & Woźniakowski [247]. The motivation is at least two-fold. One comes from the observation that integrands appearing in applications are often such that they vary more in some coordinates than in others and hence not all variables are of equal importance for the integration problem. The second one comes from the bounds on the various discrepancies. Here we introduce the first motivation, the second motivation is given in Section 3.6. In the following we use toy examples which highlight the features we are after (but

which are not directly appearing in practice as it is not obvious there that the integrand varies more in some coordinates than in others).

An extreme example of a function varying more in one coordinate than in another would be $f : [0,1]^2 \to \mathbb{R}$ given by $f(x_1, x_2) = g(x_1)$, with $g : [0,1] \to \mathbb{R}$. This function does not depend on the second variable x_2 altogether, so although it is defined as a two-dimensional function, it is, as far as numerical integration is concerned, only a one-dimensional function. Or, less extreme, we can have a function $f(x_1, x_2) = f_1(x_1) + f_2(x_2)$, with $f_1, f_2 : [0,1] \to \mathbb{R}$. In this case we can apply the same rule to the first and second coordinate simultaneously, i.e. $Q_N(f) = \frac{1}{N}\sum_{n=0}^{N-1} f(x_n, x_n) = \frac{1}{N}\sum_{n=0}^{N-1} f_1(x_n) + \frac{1}{N}\sum_{n=0}^{N-1} f_2(x_n)$. Again, as far as numerical integration is concerned, a one-dimensional rule would be sufficient.

More generally, we can have $f(\boldsymbol{x}) = \sum_{\mathfrak{u}\subseteq\mathcal{I}_s} f_{\mathfrak{u}}(\boldsymbol{x}_{\mathfrak{u}})$, where $f_{\mathfrak{u}}$ depends only on x_i for which $i \in \mathfrak{u}$, (this representation is of course not unique) and where for some $\mathfrak{u}$ we may have $f_{\mathfrak{u}} = 0$, such that $I(f) = \sum_{\mathfrak{u}\subseteq\mathcal{I}_s} I(f_{\mathfrak{u}})$. In general we might not directly have $f_{\mathfrak{u}} = 0$, but something "small" (for the purpose of numerical integration). In this case our QMC rule does not need to have "good" projections onto the coordinates in $\mathfrak{u}$ if the contribution of $f_{\mathfrak{u}}$ to the value of the integral $\int_{[0,1]^s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$ is negligible. That is, we do not need to pay much attention to obtain good accuracy in approximating $I(f_{\mathfrak{u}})$ by $Q_N(f_{\mathfrak{u}})$, which allows us to focus more on the important projections.

In order to account for that, we want such properties to be reflected in the reproducing kernel Hilbert spaces and thus also in the criterion for assessing the quality of the quadrature points. This leads to weighted reproducing kernel Hilbert spaces originating from [247].

In the following we introduce a decomposition $f(\boldsymbol{x}) = \sum_{\mathfrak{u}\subseteq\mathcal{I}_s} f_{\mathfrak{u}}(\boldsymbol{x}_{\mathfrak{u}})$, which has some further useful properties. These properties are then used to introduce weighted reproducing kernel Hilbert spaces.

Orthogonal decomposition of a reproducing kernel Hilbert space

As an example we first consider the Hilbert space $\mathscr{H}$ of absolutely continuous functions $f : [0,1] \to \mathbb{R}$ whose first derivative is square integrable. The inner product in $\mathscr{H}$ is given by

$$\langle f, g\rangle = \int_0^1 f(y)\,\mathrm{d}y \int_0^1 g(y)\,\mathrm{d}y + \int_0^1 f'(y)g'(y)\,\mathrm{d}y.$$

From the inner product one can see that constant functions are orthogonal to functions which integrate to 0, i.e., for $f(x) = c$, with $c \in \mathbb{R}$ a constant

and hence $f' = 0$, and a function $g \in \mathscr{H}$ with $\int_0^1 g(y)\,\mathrm{d}y = 0$ we always have $\langle f, g\rangle = 0$.

On the other hand, every function $f \in \mathscr{H}$ can be written as $f(x) = c + g(x)$ such that $g \in \mathscr{H}$ with $\int_0^1 g(y)\,\mathrm{d}y = 0$. Thus if we set $\mathscr{H}_1 = \{f = c\}$, the set of all constant functions in $\mathscr{H}$, and $\mathscr{H}_2 = \{g \in \mathscr{H} : \int_0^1 g(y)\,\mathrm{d}y = 0\}$, we obtain an orthogonal decomposition of $\mathscr{H}$: $\langle f_1, f_2\rangle = 0$ for all $f_1 \in \mathscr{H}_1$, $f_2 \in \mathscr{H}_2$ and for every $f \in \mathscr{H}$ there are unique functions $f_1 \in \mathscr{H}_1$, $f_2 \in \mathscr{H}_2$ such that $f = f_1 + f_2$. Indeed, for a given $f \in \mathscr{H}$, we set $f_1 := \int_0^1 f(y)\,\mathrm{d}y$ and $f_2 := f - f_1$, then $f_1 \in \mathscr{H}_1$ and $f_2 \in \mathscr{H}$ with $\int_0^1 f_2(y)\,\mathrm{d}y = \int_0^1 f(y)\,\mathrm{d}y - f_1 = 0$, and so $f_2 \in \mathscr{H}_2$.

It can be checked that (see Exercise 2.11) the reproducing kernel for $\mathscr{H}$ is given by

$$K(x, y) = 1 + B_1(x)B_1(y) + \frac{B_2(|x - y|)}{2},$$

where $B_1(t) = t - 1/2$ and $B_2(t) = t^2 - t + 1/6$ (B_1 is the first and B_2 is the second Bernoulli polynomial). Hence we have

$$\langle f, K(\cdot, y)\rangle = f(y).$$

It is not too hard to see that we can obtain $f_1 = \int_0^1 f(y)\,\mathrm{d}y$ using the inner product. Indeed, there is a linear functional which maps f to f_1, and its representer is, as we have seen above, $\int_0^1 K(x, y)\,\mathrm{d}y = 1$. Thus

$$\langle f, 1\rangle = \int_0^1 f(y)\,\mathrm{d}y \int_0^1 1\,\mathrm{d}y + \int_0^1 f'(y)0\,\mathrm{d}y = \int_0^1 f(y)\,\mathrm{d}y.$$

Therefore we can also obtain $f_2 = f - f_1$. We have

$$f_2(y) = f(y) - f_1 = \langle f, K(\cdot, y)\rangle - \langle f, 1\rangle = \langle f, K(\cdot, y) - 1\rangle.$$

Hence we have

$$\mathscr{H}_1 = \{f_1 \in \mathscr{H} : f_1 = \langle f, 1\rangle, \text{ for some } f \in \mathscr{H}\}$$

and

$$\mathscr{H}_2 = \{f_2 \in \mathscr{H} : f_2(y) = \langle f, K(\cdot, y) - 1\rangle \ \forall y \in [0, 1], \text{ for some } f \in \mathscr{H}\}.$$

Further, $\mathscr{H}_1, \mathscr{H}_2$ are reproducing kernel Hilbert spaces themselves with inner products $\langle f, g\rangle_1 = \int_0^1 f(y)\,\mathrm{d}y \int_0^1 g(y)\,\mathrm{d}y$ and $\langle f, g\rangle_2 = \int_0^1 f'(y)g'(y)\,\mathrm{d}y$ and reproducing kernels $K_1(x, y) = 1$ and $K_2(x, y) = B_1(x)B_1(y) + B_2(|x - y|)/2$. Obviously we have $K = K_1 + K_2$. For a general result see [4].

Remark 2.20 We call the Hilbert space $\mathscr{H}$ considered in this section the *unanchored Sobolev space*. The Hilbert space with kernel $K(x, y) = 1 +$

$\min(1-x, 1-y)$ is called *anchored Sobolev space* (with anchor 1), as the inner product $\langle f, g\rangle = f(1)g(1) + \int_0^1 f'(y)g'(y)\,\mathrm{d}y$ is anchored at the point 1.

In the next section we consider now tensor products of the unanchored Sobolev space and obtain orthogonal decompositions in this case.

Unanchored Sobolev spaces over $[0,1]^s$

Let again $K(x,y) = 1 + B_1(x)B_1(y) + B_2(|x-y|)/2$ and let

$$K(\boldsymbol{x}, \boldsymbol{y}) = \prod_{i=1}^{s} K(x_i, y_i)$$

be the reproducing kernel of the s-fold tensor product of the one-dimensional unanchored Sobolev space. We call the corresponding reproducing kernel Hilbert space $\mathscr{H}_s$ with domain $[0,1]^s$ again unanchored Sobolev space. The inner product in this space is given by

$$\langle f, g\rangle \tag{2.12}$$
$$= \sum_{\mathfrak{u}\subseteq\mathcal{I}_s} \int_{[0,1]^{|\mathfrak{u}|}} \left(\int_{[0,1]^{s-|\mathfrak{u}|}} \frac{\partial^{|\mathfrak{u}|} f}{\partial \boldsymbol{x}_{\mathfrak{u}}}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}_{\mathcal{I}_s\setminus\mathfrak{u}}\right)\left(\int_{[0,1]^{s-|\mathfrak{u}|}} \frac{\partial^{|\mathfrak{u}|} g}{\partial \boldsymbol{x}_{\mathfrak{u}}}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}_{\mathcal{I}_s\setminus\mathfrak{u}}\right)\mathrm{d}\boldsymbol{x}_{\mathfrak{u}}.$$

Remark 2.21 Note that $\int_0^1 B_1(y)\,\mathrm{d}y = \int_0^1 B_2(y)\,\mathrm{d}y = 0$ (see Exercise 2.18). Since $B_2(y) = B_2(1-y)$ we have $\int_0^1 B_2(|x-y|)\,\mathrm{d}y = \int_0^x B_2(x-y)\,\mathrm{d}y + \int_x^1 B_2(1-(y-x))\,\mathrm{d}y = \int_0^1 B_2(z)\,\mathrm{d}z = 0$. Altogether we obtain that $\int_0^1 K(x,y)\,\mathrm{d}y = 1$.

Recall that for $f \in \mathscr{H}_s$ we want to have a decomposition of the form

$$f(\boldsymbol{x}) = \sum_{\mathfrak{u}\subseteq\mathcal{I}_s} f_{\mathfrak{u}}(\boldsymbol{x}_{\mathfrak{u}}),$$

where $f_{\mathfrak{u}}$ only depends on the variables x_i for $i \in \mathfrak{u}$. From the previous section we know that we can decompose a one-variable function into a constant part and a variable part. We can now apply this same procedure to each of the s variables of f to decompose it into functions $f_{\mathfrak{u}}$ which depend only on the variables x_i for which $i \in \mathfrak{u}$. For $i \notin \mathfrak{u}$ the function $f_{\mathfrak{u}}$ is constant with respect to x_i, i.e. does not depend on x_i.

For $\mathfrak{u} \subseteq \mathcal{I}_s$ let

$$K_{\mathfrak{u}}(\boldsymbol{x}_{\mathfrak{u}}, \boldsymbol{y}_{\mathfrak{u}}) = \prod_{i\in\mathfrak{u}} (B_1(x_i)B_1(y_i) + B_2(|x_i - y_i|)/2),$$

where $K_\emptyset = 1$. We write $K_\mathfrak{u}(\boldsymbol{x}_\mathfrak{u}, \boldsymbol{y}_\mathfrak{u})$ and $K_\mathfrak{u}(\boldsymbol{x}, \boldsymbol{y})$ interchangeably. Then $K(\boldsymbol{x}, \boldsymbol{y}) = \sum_{\mathfrak{u} \subseteq \mathcal{I}_s} K_\mathfrak{u}(\boldsymbol{x}_\mathfrak{u}, \boldsymbol{y}_\mathfrak{u})$.

Let now

$$\mathscr{H}_\mathfrak{u} = \{f_\mathfrak{u} \in \mathscr{H}_s \,:\, f_\mathfrak{u}(\boldsymbol{y}) := \langle f, K_\mathfrak{u}(\cdot, \boldsymbol{y})\rangle \ \forall \boldsymbol{y} \in [0,1]^s \text{ for some } f \in \mathscr{H}_s\}.$$

Then for $i \in \mathfrak{u}$ we have

$$\int_0^1 f_\mathfrak{u}(\boldsymbol{y})\,\mathrm{d}y_i = \int_0^1 \langle f, K_\mathfrak{u}(\cdot, \boldsymbol{y})\rangle\,\mathrm{d}y_i = \left\langle f, \int_0^1 K_\mathfrak{u}(\cdot, \boldsymbol{y})\,\mathrm{d}y_i \right\rangle = \langle f, 0\rangle = 0,$$

as $\int_0^1 K(x,y)\,\mathrm{d}y = 1$ according to Remark 2.21. Further, by definition, $f_\mathfrak{u}$ does not depend on variables y_i for $i \notin \mathfrak{u}$ and thus $\frac{\partial f_\mathfrak{u}}{\partial y_i} = 0$ for $i \notin \mathfrak{u}$. For $f_\mathfrak{u} \in \mathscr{H}_\mathfrak{u}$ we often write $f_\mathfrak{u}(\boldsymbol{y}_\mathfrak{u})$ instead of $f_\mathfrak{u}(\boldsymbol{y})$, to emphasise that $f_\mathfrak{u}$ only depends on y_i for $i \in \mathfrak{u}$.

On the other hand, if $f \in \mathscr{H}_s$ with $\int_0^1 f(\boldsymbol{x})\,\mathrm{d}x_i = 0$ for $i \in \mathfrak{u}$ and $\frac{\partial f}{\partial x_i} = 0$ for $i \notin \mathfrak{u}$ and $g \in \mathscr{H}_s$, then

$$\begin{aligned}
&\langle f, g\rangle \\
&= \sum_{\mathfrak{v} \subseteq \mathcal{I}_s} \int_{[0,1]^{|\mathfrak{v}|}} \left(\int_{[0,1]^{s-|\mathfrak{v}|}} \frac{\partial^{|\mathfrak{v}|} f}{\partial \boldsymbol{x}_\mathfrak{v}}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}_{\mathcal{I}_s\setminus\mathfrak{v}}\right)\left(\int_{[0,1]^{s-|\mathfrak{v}|}} \frac{\partial^{|\mathfrak{v}|} g}{\partial \boldsymbol{x}_\mathfrak{v}}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}_{\mathcal{I}_s\setminus\mathfrak{v}}\right)\mathrm{d}\boldsymbol{x}_\mathfrak{v} \\
&= \int_{[0,1]^{|\mathfrak{u}|}} \left(\int_{[0,1]^{s-|\mathfrak{u}|}} \frac{\partial^{|\mathfrak{u}|} f}{\partial \boldsymbol{x}_\mathfrak{u}}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}_{\mathcal{I}_s\setminus\mathfrak{u}}\right)\left(\int_{[0,1]^{s-|\mathfrak{u}|}} \frac{\partial^{|\mathfrak{u}|} g}{\partial \boldsymbol{x}_\mathfrak{u}}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}_{\mathcal{I}_s\setminus\mathfrak{u}}\right)\mathrm{d}\boldsymbol{x}_\mathfrak{u} \\
&=: \langle f, g\rangle_\mathfrak{u},
\end{aligned}$$

as we have $\int_{[0,1]^{s-|\mathfrak{v}|}} \frac{\partial^{|\mathfrak{v}|} f}{\partial \boldsymbol{x}_\mathfrak{v}}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}_{\mathcal{I}_s\setminus\mathfrak{v}} = \frac{\partial^{|\mathfrak{v}|}}{\partial \boldsymbol{x}_\mathfrak{v}} \int_{[0,1]^{s-|\mathfrak{v}|}} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}_{\mathcal{I}_s\setminus\mathfrak{v}}$ and therefore $\int_0^1 f(\boldsymbol{x})\,\mathrm{d}x_i = 0$ for $i \in \mathfrak{u}$ and $\frac{\partial f}{\partial x_i} = 0$ for $i \notin \mathfrak{u}$ imply that we obtain $\int_{[0,1]^{s-|\mathfrak{v}|}} \frac{\partial^{|\mathfrak{v}|} f}{\partial \boldsymbol{x}_\mathfrak{v}}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}_{\mathcal{I}_s\setminus\mathfrak{v}} = 0$ for $\mathfrak{v} \neq \mathfrak{u}$. (That the order of integration and differentiation can be changed can be seen in the following way: As the order can be changed for the reproducing kernel it follows that $\langle\cdot,\cdot\rangle'$ given by

$$\begin{aligned}
\langle f, g\rangle' = \sum_{\mathfrak{u} \subseteq \mathcal{I}_s} \int_{[0,1]^{|\mathfrak{u}|}} &\frac{\partial^{|\mathfrak{u}|}}{\partial \boldsymbol{x}_\mathfrak{u}} \left(\int_{[0,1]^{s-|\mathfrak{u}|}} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}_{\mathcal{I}_s\setminus\mathfrak{u}}\right) \\
&\times \frac{\partial^{|\mathfrak{u}|}}{\partial \boldsymbol{x}_\mathfrak{u}} \left(\int_{[0,1]^{s-|\mathfrak{u}|}} g(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}_{\mathcal{I}_s\setminus\mathfrak{u}}\right) \mathrm{d}\boldsymbol{x}_\mathfrak{u},
\end{aligned}$$

is also an inner product in $\mathscr{H}_s$. From [4] we know that the inner product is unique and hence $\langle f, g\rangle = \langle f, g\rangle'$ for all $f, g \in \mathscr{H}_s$. Therefore we can change the order of integration and differentiation.)

For such a function f we then have

$$f(\boldsymbol{y}) = \langle f, K(\cdot, \boldsymbol{y})\rangle = \langle f, K_{\mathfrak{u}}(\cdot, \boldsymbol{y}_{\mathfrak{u}})\rangle = \langle f, K_{\mathfrak{u}}(\cdot, \boldsymbol{y}_{\mathfrak{u}})\rangle_{\mathfrak{u}}.$$

Thus

$$\mathscr{H}_{\mathfrak{u}} = \left\{ f \in \mathscr{H} : \int_0^1 f(\boldsymbol{x})\,\mathrm{d}x_i = 0 \text{ for } i \in \mathfrak{u} \text{ and } \frac{\partial f}{\partial x_i} = 0 \text{ for } i \notin \mathfrak{u} \right\}$$

and for $f, g \in \mathscr{H}_{\mathfrak{u}}$ we have the inner product

$$\begin{aligned} &\langle f, g\rangle_{\mathfrak{u}} \qquad (2.13)\\ &= \int_{[0,1]^{|\mathfrak{u}|}} \left(\int_{[0,1]^{s-|\mathfrak{u}|}} \frac{\partial^{|\mathfrak{u}|} f}{\partial \boldsymbol{x}_{\mathfrak{u}}}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}_{\mathcal{I}_s\setminus\mathfrak{u}} \right) \left(\int_{[0,1]^{s-|\mathfrak{u}|}} \frac{\partial^{|\mathfrak{u}|} g}{\partial \boldsymbol{x}_{\mathfrak{u}}}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}_{\mathcal{I}_s\setminus\mathfrak{u}} \right) \mathrm{d}\boldsymbol{x}_{\mathfrak{u}}. \end{aligned}$$

As for $f \in \mathscr{H}_{\mathfrak{u}}$ and $g \in \mathscr{H}_{\mathcal{I}_s}$ we have $\langle f, g\rangle = \langle f, g\rangle_{\mathfrak{u}}$, $K_{\mathfrak{u}}(\cdot, \boldsymbol{y}_{\mathfrak{u}}) \in \mathscr{H}_{\mathfrak{u}}$ and $f(\boldsymbol{y}_{\mathfrak{u}}) = \langle f, K_{\mathfrak{u}}(\cdot, \boldsymbol{y}_{\mathfrak{u}})\rangle_{\mathfrak{u}}$ it follows that $K_{\mathfrak{u}}$ is the reproducing kernel for $\mathscr{H}_{\mathfrak{u}}$ with inner product $\langle\cdot,\cdot\rangle_{\mathfrak{u}}$.

Let $f \in \mathscr{H}_s$ and let again

$$f_{\mathfrak{u}}(\boldsymbol{y}) = \langle f, K_{\mathfrak{u}}(\cdot, \boldsymbol{y})\rangle.$$

Then $f_{\mathfrak{u}} \in \mathscr{H}_{\mathfrak{u}}$ and we have

$$\sum_{\mathfrak{u}\subseteq\mathcal{I}_s} f_{\mathfrak{u}}(\boldsymbol{y}_{\mathfrak{u}}) = \sum_{\mathfrak{u}\subseteq\mathcal{I}_s} \langle f, K_{\mathfrak{u}}(\cdot, \boldsymbol{y}_{\mathfrak{u}})\rangle = \left\langle f, \sum_{\mathfrak{u}\subseteq\mathcal{I}_s} K_{\mathfrak{u}}(\cdot, \boldsymbol{y}_{\mathfrak{u}}) \right\rangle = \langle f, K(\cdot, \boldsymbol{y})\rangle = f(\boldsymbol{y}).$$

Further, for $f, g \in \mathscr{H}_s$ we also have

$$\langle f, g\rangle = \sum_{\mathfrak{u}\subseteq\mathcal{I}_s} \langle f, g\rangle_{\mathfrak{u}} = \sum_{\mathfrak{u}\subseteq\mathcal{I}_s} \langle f_{\mathfrak{u}}, g_{\mathfrak{u}}\rangle_{\mathfrak{u}}.$$

The first equality follows from (2.12) and (2.13) and the second equality follows as, for $\mathfrak{v} \neq \mathfrak{u}$ and $f_{\mathfrak{v}} \in \mathscr{H}_{\mathfrak{v}}$, we have $\langle f_{\mathfrak{v}}, g\rangle_{\mathfrak{u}} = 0$ for all $g \in \mathscr{H}_{\mathfrak{u}}$. In particular we have

$$\|f\|^2 = \sum_{\mathfrak{u}\subseteq\mathcal{I}_s} \|f_{\mathfrak{u}}\|_{\mathfrak{u}}^2. \qquad (2.14)$$

ANOVA decomposition

The orthogonal decomposition of $\mathscr{H}_s$ considered in the previous section has some other interesting properties, as shown in [54, Section 6]. The functions $f_{\mathfrak{u}}$ can also be found in an inductive way by using the properties

$\int_0^1 f_{\mathfrak{u}}(\boldsymbol{x})\,\mathrm{d}x_i = 0$ for $i \in \mathfrak{u}$ and $\int_0^1 f_{\mathfrak{u}}(\boldsymbol{x})\,\mathrm{d}x_i = f_{\mathfrak{u}}(\boldsymbol{x})$ for $i \notin \mathfrak{u}$ ($f_{\mathfrak{u}}$ does not depend on x_i for $i \notin \mathfrak{u}$). Thus

$$\int_{[0,1]^{s-|\mathfrak{u}|}} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}_{\mathcal{I}_s\setminus\mathfrak{u}} = \sum_{\mathfrak{v}\subseteq\mathcal{I}_s} \int_{[0,1]^{s-|\mathfrak{u}|}} f_{\mathfrak{v}}(\boldsymbol{x}_{\mathfrak{v}})\,\mathrm{d}\boldsymbol{x}_{\mathcal{I}_s\setminus\mathfrak{u}} = \sum_{\mathfrak{v}\subseteq\mathfrak{u}} f_{\mathfrak{v}}(\boldsymbol{x}_{\mathfrak{v}}).$$

Starting with $\mathfrak{u} = \emptyset$ we can obtain the functions $f_{\mathfrak{u}}$ inductively: $f_\emptyset = \int_{[0,1]^s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$ and if $f_{\mathfrak{v}}$ is known for all $\mathfrak{v} \subset \mathfrak{u}$ we obtain

$$f_{\mathfrak{u}}(\boldsymbol{x}_{\mathfrak{u}}) = \int_{[0,1]^{s-|\mathfrak{u}|}} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}_{\mathcal{I}_s\setminus\mathfrak{u}} - \sum_{\mathfrak{v}\subset\mathfrak{u}} f_{\mathfrak{v}}(\boldsymbol{x}_{\mathfrak{v}}).$$

Example 2.22 Let $f(x_1, x_2) = \mathrm{e}^{x_1} - x_2 + x_1 \sin(\pi x_2)$. Then

$$f_\emptyset = \int_0^1 \int_0^1 f(x_1, x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 = \mathrm{e}{-}1{-}1/2{+}\frac{1}{2\pi}(\cos(0){-}\cos(\pi)) = \mathrm{e}{-}3/2{+}\frac{1}{\pi}.$$

Now we can calculate $f_{\{1\}}$ and $f_{\{2\}}$, we have

$$f_{\{1\}}(x_1) = \int_0^1 f(x_1, x_2)\,\mathrm{d}x_2 - f_\emptyset = \mathrm{e}^{x_1} + \frac{2x_1}{\pi} - \mathrm{e} + 1 - \frac{1}{\pi}$$

and

$$f_{\{2\}}(x_2) = \int_0^1 f(x_1, x_2)\,\mathrm{d}x_1 - f_\emptyset = -x_2 + \frac{1}{2}\sin(\pi x_2) + \frac{1}{2} - \frac{1}{\pi}.$$

Finally we can calculate $f_{\{1,2\}}$, we have

$$\begin{aligned} f_{\{1,2\}}(x_1, x_2) &= f(x_1, x_2) - f_{\{1\}}(x_1) - f_{\{2\}}(x_2) - f_\emptyset \\ &= (x_1 - 1/2)\sin(\pi x_2) + \frac{1 - 2x_1}{\pi}. \end{aligned}$$

The variance Var of a function f is given by $\mathrm{Var}(f) = \int_{[0,1]^s} f^2(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \left(\int_{[0,1]^s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right)^2$. Using the decomposition of f we obtain

$$\begin{aligned} \mathrm{Var}(f) &= \int_{[0,1]^s} f^2(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \left(\int_{[0,1]^s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right)^2 \\ &= \sum_{\emptyset\neq\mathfrak{u},\mathfrak{v}\subseteq\mathcal{I}_s} \int_{[0,1]^s} f_{\mathfrak{u}}(\boldsymbol{x}_{\mathfrak{u}}) f_{\mathfrak{v}}(\boldsymbol{x}_{\mathfrak{v}})\,\mathrm{d}\boldsymbol{x}. \end{aligned}$$

Using the fact that $\int_0^1 f_{\mathfrak{u}}(\boldsymbol{x}_{\mathfrak{u}})\,\mathrm{d}x_i = 0$ for $i \in \mathfrak{u}$ we obtain $\int_{[0,1]^s} f_{\mathfrak{u}}(\boldsymbol{x}_{\mathfrak{u}}) f_{\mathfrak{v}}(\boldsymbol{x}_{\mathfrak{v}})\,\mathrm{d}\boldsymbol{x} = 0$ for $\mathfrak{u} \neq \mathfrak{v}$. Further $\int_{[0,1]^s} f_{\mathfrak{u}}(\boldsymbol{x}_{\mathfrak{u}}) f_{\mathfrak{u}}(\boldsymbol{x}_{\mathfrak{u}})\,\mathrm{d}\boldsymbol{x} = \int_{[0,1]^{|\mathfrak{u}|}} f_{\mathfrak{u}}^2(\boldsymbol{x}_{\mathfrak{u}})\,\mathrm{d}\boldsymbol{x}_{\mathfrak{u}} = \mathrm{Var}(f_{\mathfrak{u}})$

since $\int_{[0,1]^s} f_{\mathfrak{u}}(\boldsymbol{x}_{\mathfrak{u}})\,\mathrm{d}\boldsymbol{x}_{\mathfrak{u}} = 0$. Therefore and since $\mathrm{Var}(f_\emptyset) = 0$ we obtain

$$\mathrm{Var}(f) = \sum_{\mathfrak{u}\subseteq\mathcal{I}_s} \mathrm{Var}(f_{\mathfrak{u}}). \tag{2.15}$$

ANOVA decomposition stands for *ANalysis Of VAriance*. The decomposition of f as used above can be used to analyse the contributions of lower dimensional projections $f_{\mathfrak{u}}$ to the total variance via the formula $\mathrm{Var}(f) = \sum_{\mathfrak{u}\subseteq\mathcal{I}_s} \mathrm{Var}(f_{\mathfrak{u}})$.

Example 2.23 We calculate the variances of $f, f_{\{1\}}, f_{\{2\}}, f_{\{1,2\}}$ from Example 2.22. We have

$$\begin{aligned}
\mathrm{Var}(f) &= \frac{-4 - 8\,e^1\pi + 24\,\pi - 2\,\pi^2 e^2 - 5\,\pi^2 + 8\,e^1\pi^2}{4\pi^2},\\
\mathrm{Var}(f_{\{1\}}) &= \frac{12\,e^1\pi^2 + 36\,\pi - 9\,\pi^2 - 3\,\pi^2 e^2 + 2 - 12\,e^1\pi}{6\pi^2},\\
\mathrm{Var}(f_{\{2\}}) &= \frac{-24 + 5\,\pi^2}{24\pi^2},\\
\mathrm{Var}(f_{\{1,2\}} &= \frac{-8 + \pi^2}{24\pi^2},
\end{aligned}$$

and therefore $\mathrm{Var}(f) = \mathrm{Var}(f_{\{1\}}) + \mathrm{Var}(f_{\{2\}}) + \mathrm{Var}(f_{\{1,2\}})$.

Weighted reproducing kernel Hilbert spaces

Equation (2.14) now holds the key to weighted reproducing kernel Hilbert spaces. Recall that for the worst-case error we consider all functions in the unit ball of the space, i.e. all $f \in \mathscr{H}_s$ with $\|f\| \le 1$. Using (2.14) this amounts to $\sum_{\mathfrak{u}\subseteq\mathcal{I}_s} \|f_{\mathfrak{u}}\|_{\mathfrak{u}}^2 \le 1$, where $f(\boldsymbol{x}) = \sum_{\mathfrak{u}\subseteq\mathcal{I}_s} f_{\mathfrak{u}}(\boldsymbol{x}_{\mathfrak{u}})$.

The worst-case error is used as a criterion for choosing the quadrature points. By a small change to the norm we can change the shape of the unit ball considered in the worst-case error, and thereby also the criterion used for measuring the quality of quadrature points.

It has been observed that many integrands from applications seem to vary more in lower dimensional projections than higher dimensional ones. We model this behaviour now in the following way: We can write $f(\boldsymbol{x}) = \sum_{\mathfrak{u}\subseteq\mathcal{I}_s} f_{\mathfrak{u}}(\boldsymbol{x}_{\mathfrak{u}})$. Some of the $f_{\mathfrak{u}}$ are "small", which we can now make more precise by saying that $\|f_{\mathfrak{u}}\|_{\mathfrak{u}}$ is small, compared with the norm of other projections. In order to change the unit ball such that only functions for which $\|f_{\mathfrak{u}}\|_{\mathfrak{u}}$ is small are contained in it, we multiply $\|f_{\mathfrak{u}}\|_{\mathfrak{u}}$ by a real number

$\gamma_{\mathfrak{u}}$. Let $\boldsymbol{\gamma} := \{\gamma_{\mathfrak{u}} : \mathfrak{u} \subseteq \mathcal{I}_s\}$. Then we define a new "weighted" norm by

$$\|f\|_{\boldsymbol{\gamma}}^2 = \sum_{\mathfrak{u} \subseteq \mathcal{I}_s} \gamma_{\mathfrak{u}}^{-1} \|f_{\mathfrak{u}}\|_{\mathfrak{u}}^2.$$

Then the condition $\|f\|_{\boldsymbol{\gamma}} \leq 1$ in the definition of the worst-case error (Definition 2.10) implies that if $\gamma_{\mathfrak{u}}$ is small, then also $\|f_{\mathfrak{u}}\|$ has to be small in order for f to satisfy $\|f\|_{\boldsymbol{\gamma}} \leq 1$. The corresponding inner product then has the form

$$\langle f, g \rangle_{\boldsymbol{\gamma}} = \sum_{\mathfrak{u} \subseteq \mathcal{I}_s} \gamma_{\mathfrak{u}}^{-1} \langle f, g \rangle_{\mathfrak{u}}. \tag{2.16}$$

We now work out how this modification affects the theory which we established until now. The Hilbert space $\mathscr{H}_{\boldsymbol{\gamma}}$ with inner product (2.16) is a reproducing kernel Hilbert space with reproducing kernel

$$K_{\boldsymbol{\gamma}}(\boldsymbol{x}, \boldsymbol{y}) = \sum_{\mathfrak{u} \subseteq \mathcal{I}_s} \gamma_{\mathfrak{u}} K_{\mathfrak{u}}(\boldsymbol{x}_{\mathfrak{u}}, \boldsymbol{y}_{\mathfrak{u}}).$$

Indeed we have $K_{\boldsymbol{\gamma}}(\cdot, \boldsymbol{y}) \in \mathscr{H}_{\boldsymbol{\gamma}}$ and

$$\langle f, K_{\boldsymbol{\gamma}}(\cdot, \boldsymbol{y}) \rangle_{\boldsymbol{\gamma}} = \sum_{\mathfrak{u} \subseteq \mathcal{I}_s} \gamma_{\mathfrak{u}}^{-1} \langle f, \gamma_{\mathfrak{u}} K_{\mathfrak{u}}(\cdot, \boldsymbol{y}_{\mathfrak{u}}) \rangle_{\mathfrak{u}} = \sum_{\mathfrak{u} \subseteq \mathcal{I}_s} \langle f, K_{\mathfrak{u}}(\cdot, \boldsymbol{y}_{\mathfrak{u}}) \rangle_{\mathfrak{u}} = f(\boldsymbol{y}).$$

Using Proposition 2.11 we obtain the weighted square worst-case error

$$\begin{aligned} e^2(\mathscr{H}_{\boldsymbol{\gamma}}, \mathcal{P}) &= \frac{1}{N^2} \sum_{m,n=0}^{N-1} \sum_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{I}_s} \gamma_{\mathfrak{u}} K_{\mathfrak{u}}(\boldsymbol{x}_m, \boldsymbol{x}_n) \\ &= \frac{1}{N^2} \sum_{m,n=0}^{N-1} \sum_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{I}_s} \gamma_{\mathfrak{u}} \prod_{i \in \mathfrak{u}} (B_1(x_{m,i}) B_1(x_{n,i}) + B_2(|x_{m,i} - x_{n,i}|)/2) \\ &= \sum_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{I}_s} \gamma_{\mathfrak{u}} \frac{1}{N^2} \sum_{m,n=0}^{N-1} \prod_{i \in \mathfrak{u}} (B_1(x_{m,i}) B_1(x_{n,i}) + B_2(|x_{m,i} - x_{n,i}|)/2). \end{aligned}$$

The worst-case error is a measure for the quality of the quadrature points. Observe that $\frac{1}{N^2} \sum_{m,n=0}^{N-1} \prod_{i \in \mathfrak{u}} (B_1(x_{m,i}) B_1(x_{n,i}) + B_2(|x_{m,i} - x_{n,i}|)/2)$ is the worst-case error for the reproducing kernel Hilbert space $\mathscr{H}_{\mathfrak{u}}$ and hence measures the quality of the projection of the quadrature points onto the coordinates in $\mathfrak{u}$.

Recall that if for some $\emptyset \neq \mathfrak{u} \subseteq \mathcal{I}_s$ the value of $\gamma_{\mathfrak{u}}$ is small, than $\|f_{\mathfrak{u}}\|_{\mathfrak{u}}$ also has to be small. On the other hand, if $\gamma_{\mathfrak{u}}$ is small (compared to $\gamma_{\mathfrak{v}}$ for $\mathfrak{v} \neq \mathfrak{u}$), then $\gamma_{\mathfrak{u}} \frac{1}{N^2} \sum_{m,n=0}^{N-1} \prod_{i \in \mathfrak{u}} (B_1(x_{m,i}) B_1(x_{n,i}) + B_2(|x_{m,i} - x_{n,i}|)/2)$ is also small, regardless of whether $\frac{1}{N^2} \sum_{m,n=0}^{N-1} \prod_{i \in \mathfrak{u}} (B_1(x_{m,i}) B_1(x_{n,i}) + B_2(|x_{m,i} -$

$x_{n,i}|)/2)$ is large or small. This makes sense, since $\|f_{\mathrm{u}}\|$ small means we do not need to focus on approximating the integral $\int_{[0,1]^{|\mathrm{u}|}} f_{\mathrm{u}}(\boldsymbol{x}_{\mathrm{u}})\,\mathrm{d}\boldsymbol{x}_{\mathrm{u}}$ and so the quality of the approximation does not matter much.

In the next chapter we look more closely at geometrical properties of the discrepancy.

Exercises

2.1 Calculate the star discrepancy and the L_2-discrepancy of the point sets

- $\mathcal{P}_1 = \{\frac{n}{N} : 0 \le n \le N-1\}$,
- $\mathcal{P}_2 = \{\frac{2n+1}{2N} : 0 \le n \le N-1\}$,
- $\mathcal{P}_3 = \{\frac{n}{2N} : 0 \le n \le N-1\}$.

2.2 Let $P = \{p(x) = a_0 + a_1 x + \cdots + a_r x^r : a_0, \ldots, a_r \in \mathbb{R}\}$ be the space of all polynomials of degree at most r. We define an inner product on this space by: for $p(x) = a_0 + a_1 x + \cdots + a_r x^r$ and $q(x) = b_0 + b_1 x + \cdots + b_r x^r$ let $\langle p, q\rangle = a_0 b_0 + a_1 b_1 + \cdots + a_r b_r$. What is the reproducing kernel for this space? Prove properties P1 - P5 for this kernel.

2.3 Let $P = \{f(x) = a_0 + a_1 \mathrm{e}^{2\pi \mathrm{i} x} + \cdots + a_r \mathrm{e}^{2\pi \mathrm{i} r x} : a_0, a_1, \ldots, a_r \in \mathbb{C}\}$ be the space of all trigonometric polynomials of degree at most r. We define an inner product on this space by: for $f(x) = a_0 + a_1 \mathrm{e}^{2\pi \mathrm{i} x} + \cdots + a_r \mathrm{e}^{2\pi \mathrm{i} r x}$ and $g(x) = b_0 + b_1 \mathrm{e}^{2\pi \mathrm{i} x} + \cdots + b_r \mathrm{e}^{2\pi \mathrm{i} r x}$ let $\langle f, g\rangle = a_0 b_0 + a_1 b_1 + \cdots + a_r b_r$. What is the reproducing kernel for this space? Prove properties P1 - P5 for this kernel.

2.4 The one-dimensional Korobov space $\mathscr{H}_{\mathrm{kor},\alpha}$ for real $\alpha > 1$ consists of all one-periodic L_1-functions $f : [0,1] \to \mathbb{C}$ with absolute convergent Fourier series representation such that $|\widehat{f}(\boldsymbol{h})| = O(\max(1, |h|^{\alpha}))$ for integers h. The reproducing kernel for the Korobov space is given by $K(x,y) = 1 + \sum_{h \in \mathbb{Z}, h \neq 0} |h|^{-\alpha} \mathrm{e}^{2\pi \mathrm{i} h(x-y)}$. What is the inner product for this space?

2.5 Verify Remark 2.6, by showing that $K(x,y) := \min(1-x, 1-y)$ satisfies P1-P5 for a suitable inner product.

2.6 Verify that $\Delta_{\mathcal{P}}(y) = \frac{\mathrm{d}}{\mathrm{d}y}\left(\int_0^1 K(x,y)\,\mathrm{d}x - \frac{1}{N}\sum_{n=0}^{N-1} K(x_n, y)\right)$, where $\Delta_{\mathcal{P}}$ is the discrepancy function and $K(x,y) = 1 + \min(1-x, 1-y)$.

2.7 Let $\mathscr{H}_{\mathrm{wal}}$ be the Walsh space as defined in Example 2.8. Show that the worst-case error for a QMC rule using a point set $\mathcal{P} = \{x_0, \ldots, x_{N-1}\}$ is given by

$$e^2(\mathscr{H}_{\mathrm{wal}}, \mathcal{P}) = \sum_{k=1}^{\infty} r_{\mathrm{wal},b,\alpha}(k) \left| \frac{1}{N} \sum_{n=0}^{N-1} {}_b\mathrm{wal}_k(x_n) \right|^2.$$

Hint: See Appendix A for more information on Walsh functions; see [50, Section 2 and 4] for more information on the reproducing kernel Hilbert space generated by K and numerical integration therein.

2.8 Evaluate the integral $\int_{[0,1]^s} |\Delta_{\mathcal{P}}(\boldsymbol{x})|^2 \,\mathrm{d}\boldsymbol{x}$ to obtain Proposition 2.15.

2.9 For $s = 2$ prove Proposition 2.17 directly by evaluating the integrals on the right hand side of the formula.

2.10 Study the proof of the classical Koksma-Hlawka inequality in [128, p. 143–153].

2.11 Check that for $K(x, y) = 1 + B_1(x)B_1(y) + B_2(|x-y|)/2$, with $B_1(t) = t - 1/2$ and $B_2(t) = t^2 - t + 1/6$ and an inner product $\langle f, g\rangle = \int_0^1 f(y)\,\mathrm{d}y \int_0^1 g(y)\,\mathrm{d}y + \int_0^1 f'(y)g'(y)\,\mathrm{d}y$ we always have $\langle f, K(\cdot, y)\rangle = f(y)$ and thus conclude that K is the reproducing kernel of the Hilbert space of absolutely continuous functions with square integrable first derivative.

2.12 Obtain a Warnock type formula, Hlawka identity and Koksma-Hlawka inequality for the reproducing kernel

$$K(\boldsymbol{x}, \boldsymbol{y}) = \prod_{i=1}^{s} (1 + B_1(x_i)B_1(y_i) + B_2(|x_i - y_i|)/2),$$

where $B_1(t) = t - 1/2$ and $B_2(t) = t^2 - t + 1/6$ (this is the kernel considered in Section 2.5). The inner product in the associated reproducing kernel Hilbert space is given by

$$\langle f, g\rangle = \sum_{\mathfrak{u} \subseteq \mathcal{I}_s} \int_{[0,1]^{|\mathfrak{u}|}} \left(\int_{[0,1]^{s-|\mathfrak{u}|}} \frac{\partial^{|\mathfrak{u}|} f}{\partial \boldsymbol{x}_{\mathfrak{u}}}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}_{\mathcal{I}_s \setminus \mathfrak{u}} \right) \times \left(\int_{[0,1]^{s-|\mathfrak{u}|}} \frac{\partial^{|\mathfrak{u}|} g}{\partial \boldsymbol{x}_{\mathfrak{u}}}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}_{\mathcal{I}_s \setminus \mathfrak{u}} \right) \mathrm{d}\boldsymbol{x}_{\mathfrak{u}}.$$

(Note that the discrepancy function differs from the discrepancy function in Definition 2.13 in this case.) See [249] for more information on this space.

2.13 The s-dimensional Korobov space $\mathscr{H}_{\mathrm{kor},s,\alpha}$, $s \in \mathbb{N}$ and $\alpha > 1$, is the reproducing kernel Hilbert space of complex-valued functions of period one which is defined by

$$K_\alpha(\boldsymbol{x}, \boldsymbol{y}) = \sum_{\boldsymbol{h} \in \mathbb{Z}^s} \frac{1}{r_\alpha(\boldsymbol{h})} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{h} \cdot (\boldsymbol{x} - \boldsymbol{y})},$$

where $r_\alpha(\boldsymbol{h}) = \prod_{i=1}^s (\max(1, |h_i|))^\alpha$. The inner-product is given by

$$\langle f, g\rangle_\alpha = \sum_{\boldsymbol{h}\in\mathbb{Z}^s} r_\alpha(\boldsymbol{h})\widehat{f}(\boldsymbol{h})\overline{\widehat{g}(\boldsymbol{h})}.$$

Show that the worst-case integration error for a QMC rule in $\mathscr{H}_{\mathrm{kor},s,\alpha}$ using $\mathcal{P} = \{\boldsymbol{x}_0, \dots, \boldsymbol{x}_{N-1}\}$ is given by

$$e^2(\mathscr{H}_{\mathrm{kor},s,\alpha}, \mathcal{P}) = \sum_{\boldsymbol{h}\in\mathbb{Z}^s\setminus\{\boldsymbol{0}\}} \frac{1}{r_\alpha(\boldsymbol{h})} \left|\frac{1}{N}\sum_{n=0}^{N-1} \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{h}\cdot\boldsymbol{x}_n}\right|^2.$$

2.14 Let $\widetilde{e}^2_{\alpha,N} := \int_{[0,1]^{Ns}} e^2(\mathscr{H}_{\mathrm{kor},s,\alpha}, \{\boldsymbol{x}_0, \dots, \boldsymbol{x}_{N-1}\})\,\mathrm{d}\boldsymbol{x}_0 \cdots \mathrm{d}\boldsymbol{x}_{N-1}$. Show that for $\alpha > 1$ we have

$$\widetilde{e}^2_{\alpha,N} \le \mathrm{e}^{2\zeta(\alpha)s}/N,$$

where $\zeta(\alpha) = \sum_{j=1}^\infty j^{-\alpha}$. *Hint:* See [248, Theorem 1] or [100].

2.15 Let $s \ge 1$ and $b \ge 2$ be integers, $\alpha > 1$ a real and $\boldsymbol{\gamma} = (\gamma_i)_{i\ge1}$ be a sequence of nonnegative reals. The s-dimensional weighted version of the Walsh space from Example 2.8 is the reproducing kernel Hilbert space $\mathscr{H}_{\mathrm{wal},s,b,\alpha,\boldsymbol{\gamma}}$ of b-adic Walsh series $f(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{N}_0^s} \widehat{f}(\boldsymbol{k})\,{}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x})$ with reproducing kernel defined by

$$K_{\mathrm{wal},s,b,\alpha,\boldsymbol{\gamma}}(\boldsymbol{x}, \boldsymbol{y}) = \sum_{\boldsymbol{k}\in\mathbb{N}_0^s} r_{\mathrm{wal},b,\alpha}(\boldsymbol{k}, \boldsymbol{\gamma})\,{}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x} \ominus \boldsymbol{y}),$$

where for $\boldsymbol{k} = (k_1, \dots, k_s)$ we put $r_{\mathrm{wal},b,\alpha}(\boldsymbol{k}, \boldsymbol{\gamma}) = \prod_{i=1}^s r_{\mathrm{wal},b,\alpha}(k_i, \gamma_i)$ and for $k \in \mathbb{N}_0$ and $\gamma > 0$ we write

$$r_{\mathrm{wal},b,\alpha}(k, \gamma) = \begin{cases} 1 & \text{if } k = 0, \\ \gamma b^{-\alpha a} & \text{if } k = \kappa_0 + \kappa_1 b + \cdots + \kappa_a b^a \text{ and } \kappa_a \ne 0. \end{cases}$$

The inner-product is given by

$$\langle f, g\rangle = \sum_{\boldsymbol{k}\in\mathbb{N}_0^s} r_{\mathrm{wal},b,\alpha}(\boldsymbol{k}, \boldsymbol{\gamma})^{-1}\widehat{f}(\boldsymbol{k})\overline{\widehat{g}(\boldsymbol{k})}.$$

Show that the worst-case integration error for a QMC rule in $\mathscr{H}_{\mathrm{wal},s,b,\alpha,\boldsymbol{\gamma}}$ using $\mathcal{P} = \{\boldsymbol{x}_0, \dots, \boldsymbol{x}_{N-1}\}$ is given by

$$e^2(\mathscr{H}_{\mathrm{wal},s,b,\alpha,\boldsymbol{\gamma}}, \mathcal{P}) = \sum_{\boldsymbol{k}\in\mathbb{N}_0^s\setminus\{\boldsymbol{0}\}} r_{\mathrm{wal},b,\alpha}(\boldsymbol{k}, \boldsymbol{\gamma}) \left|\frac{1}{N}\sum_{n=0}^{N-1} {}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_n)\right|^2.$$

Hint: Compare with Exercise 2.7. See [50, Section 2 and 4].

2.16 Let $\widetilde{e}^2_{n,\alpha,\boldsymbol{\gamma},N} := \int_{[0,1]^{Ns}} e^2(\mathscr{H}_{\mathrm{wal},s,b,\alpha,\boldsymbol{\gamma}}, \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\})\,\mathrm{d}\boldsymbol{x}_0 \cdots \mathrm{d}\boldsymbol{x}_{N-1}$. Show that for $\alpha > 1$ we have

$$\widetilde{e}^2_{\alpha,N} \le \mathrm{e}^{\nu_b(\alpha)\sum_{i=1}^s \gamma_i}/N,$$

where $\nu_b(\alpha) = \sum_{k=1}^\infty b^{-\alpha a(k)} = \frac{b^\alpha(b-1)}{b^\alpha - b}$ and where $a(k) = a$ whenever $k = \kappa_0 + \kappa_1 b + \cdots + \kappa_a b^a$ with $\kappa_a \ne 0$. *Hint:* See [50, Theorem 1].

2.17 Obtain an orthogonal decomposition of the reproducing kernel Hilbert space with reproducing kernel $K(x,y) = 1 + \min(1-x, 1-y)$. What are the spaces $\mathscr{H}_1, \mathscr{H}_2$, K_1, K_2 and respective inner products in this case?

2.18 Show that $\int_0^1 B_1(x)\,\mathrm{d}x = 0$ and $\int_0^1 B_2(x)\,\mathrm{d}x = 0$.

2.19 Let $f(x_1, x_2) = \mathrm{e}^{x_1 - x_2} - x_2 \cos(\pi(x_1 + x_2^2))$. Calculate the ANOVA decomposition and the variances of $f_{\mathfrak{u}}$ and check that (2.15) holds.

2.20 Using similar arguments as in Section 2.4 and 2.5, obtain a weighted version of the L_2-discrepancy, Warnock's formula, Hlawka's identity and the Koksma-Hlawka inequality for the reproducing kernel Hilbert space with reproducing kernel $K(\boldsymbol{x}, \boldsymbol{y}) = \prod_{i=1}^s (1 + \min(1-x_i, 1-y_i))$. *Hint:* See [247].

2.21 Calculate the worst-case error for numerical integration in the reproducing kernel Hilbert space $\mathscr{H}_{\mathfrak{u}}$ from Section 2.5.

3

Geometric discrepancy

In this chapter we introduce the theory of uniform distribution modulo one, for which the main motivation is the application of equidistributed points for numerical integration with QMC algorithms as we have seen in Chapter 2. The quality of the equidistribution of a point set is measured by the so-called discrepancy. We introduce different notions of discrepancy including the rather new weighted discrepancies now from the perspective of their geometrical properties. Because of their geometric interpretation these discrepancies are also often called geometric discrepancies. We provide some classical as well as new results for geometric discrepancies. A standard reference for the theory of uniform distribution modulo one is the book of Kuipers & Niederreiter [128] to which we refer for a further, more detailed discussion (mainly from a number-theoretic view point). See also the book of Drmota & Tichy [61].

3.1 Uniform distribution modulo one

As discussed in Chapter 2 we are concerned with approximating the integral of a function f over the s-dimensional unit cube $[0,1]^s$ with a QMC rule, which gives the average of function values $f(\boldsymbol{x}_n)$, where the points $\boldsymbol{x}_0, \dots, \boldsymbol{x}_{N-1}$ are deterministically chosen sample points from the unit cube, see (2.1).

For Riemann integrable functions f we would, of course, demand that for growing N the error of this approximation goes to zero, i.e., for a sequence $(\boldsymbol{x}_n)_{n\geq 0}$ in $[0,1)^s$ we would like to have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(\boldsymbol{x}_n) = \int_{[0,1]^s} f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}. \tag{3.1}$$

Hence the question arises, how the sequence of sample points has to be cho-

sen such that this is indeed the case? Let us, for a moment, assume that the function f to be integrated comes from the class of finite linear combinations of characteristic functions of axes-parallel rectangles. This is probably one of the simplest sub-class of Riemann-integrable functions. Then our question leads directly to a branch of Number Theory, namely to the Theory of Uniform Distribution Modulo One, which goes back to a fundamental work of Weyl [263] from the year 1916.

Intuitively, one may consider a sequence of points in the unit cube as uniformly distributed, if each set E from some suitable subclass of measurable sets contains (asymptotically) the right portion of points, namely $N\lambda_s(E)$, where λ_s is the s-dimensional Lebesgue measure (see Figure 3.1). This leads to the following exact definition of uniform distribution modulo one.

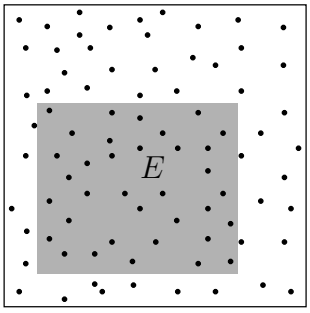

Figure 3.1 The number of points in E should be approximately $N\lambda_s(E)$.

For a sequence $\mathcal{S} = (\boldsymbol{x}_n)_{n\geq 0}$ in the s-dimensional unit cube $[0,1)^s$ and a subset E of $[0,1]^s$ let $A(E,N,\mathcal{S})$ be the number of indices n, $0 \leq n \leq N-1$, for which the point $\boldsymbol{x}_n$ belongs to E. That is, $A(E,N,\mathcal{S}) = \sum_{n=0}^{N-1} \chi_E(\boldsymbol{x}_n)$.

Definition 3.1 A sequence $\mathcal{S} = (\boldsymbol{x}_n)_{n\geq 0}$ in the s-dimensional unit cube $[0,1)^s$ is said to be *uniformly distributed modulo one*, if for every interval $[\boldsymbol{a},\boldsymbol{b}) \subseteq [0,1]^s$ we have

$$\lim_{N\to\infty} \frac{A([\boldsymbol{a},\boldsymbol{b}),N,\mathcal{S})}{N} = \lambda_s([\boldsymbol{a},\boldsymbol{b})), \tag{3.2}$$

or in other words, if (3.1) holds for the characteristic function $\chi_{[\boldsymbol{a},\boldsymbol{b})}$ of any sub-interval $[\boldsymbol{a},\boldsymbol{b}) \subseteq [0,1]^s$.

We remark that the choice of half-open intervals in the above definition and in the following is of minor importance.

Remark 3.2 There is also the stronger concept of well-distribution modulo one. For a sequence $\mathcal{S} = (\boldsymbol{x}_n)_{n\geq 0}$ in the s-dimensional unit cube $[0,1)^s$ and

a subset E of $[0,1]^s$, let $A(E,k,N,\mathcal{S})$ be the number of indices n, $k \le n \le k+N-1$, for which the point $\boldsymbol{x}_n$ belongs to E. Then the sequence $\mathcal{S}$ is said to be *well-distributed modulo one*, if for every sub-interval $[\boldsymbol{a},\boldsymbol{b}) \subseteq [0,1]^s$ we have

$$\lim_{N\to\infty} \frac{A([\boldsymbol{a},\boldsymbol{b}),k,N,\mathcal{S})}{N} = \lambda_s([\boldsymbol{a},\boldsymbol{b})) \tag{3.3}$$

uniformly in $k = 0,1,2,\ldots$.

It is obvious from the definition that a sequence $(\boldsymbol{x}_n)_{n\ge 0}$ in the s-dimensional unit cube $[0,1)^s$ is uniformly distributed modulo one, if (3.1) holds for every finite linear combination of characteristic functions of axes-parallel rectangles $f : [0,1]^s \to \mathbb{R}$.

Now it is well known from analysis that any Riemann integrable function on $[0,1]^s$ can be approximated arbitrary closely in $L_1([0,1]^s)$ by finite linear combinations of characteristic functions of axes-parallel rectangles. From this fact we obtain the following equivalence (see [128, Chapter 1, Corollary 1.1] for a more detailed proof).

Theorem 3.3 *A sequence $\mathcal{S} = (\boldsymbol{x}_n)_{n\ge 0}$ in $[0,1)^s$ is uniformly distributed modulo one, if and only if for every Riemann integrable function $f : [0,1]^s \to \mathbb{R}$ we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(\boldsymbol{x}_n) = \int_{[0,1]^s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}. \tag{3.4}$$

Note that there is no sequence such that (3.4) holds for all Lebesgue integrable functions. For a given sequence $\mathcal{S}$ with support $\overline{\mathcal{S}}$ the characteristic function of $[0,1]^s \setminus \overline{\mathcal{S}}$ is a counterexample. Furthermore, it was shown by de Bruijn & Post [31] that for every function $f \in L_1([0,1])$, which is not Riemann integrable, there exists a sequence which is uniformly distributed modulo one but for which (3.4) does not hold.

One can also show the following theorem whose proof is left as an exercise (see Exercise 3.4).

Theorem 3.4 *A sequence $(\boldsymbol{x}_n)_{n\ge 0}$ in the s-dimensional unit cube $[0,1)^s$ is uniformly distributed modulo one, if and only if* (3.4) *holds for every continuous, complex-valued function $f : [0,1]^s \to \mathbb{C}$ with period one.*

For example, let $f : [0,1]^s \to \mathbb{C}$ be given by $f(\boldsymbol{x}) = \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{h}\cdot\boldsymbol{x}}$, where $\boldsymbol{h} \in \mathbb{Z}^s$ is some s-dimensional integer vector. If a sequence $(\boldsymbol{x}_n)_{n\ge 0}$ in the s-dimensional unit cube is uniformly distributed modulo one, then by The-

orem 3.4 we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{h}\cdot\boldsymbol{x}_n} = \int_{[0,1]^s} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{h}\cdot\boldsymbol{x}} \,\mathrm{d}\boldsymbol{x},$$

where the last integral is 0 if $\boldsymbol{h} \in \mathbb{Z}^s \setminus \{\mathbf{0}\}$ and 1 if $\boldsymbol{h} = \mathbf{0}$. Astonishingly, the opposite is true as well: That is, the relation $\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{h}\cdot\boldsymbol{x}_n} = 0$ for all $\boldsymbol{h} \in \mathbb{Z}^s \setminus \{\mathbf{0}\}$ is also a sufficient condition for the sequence $(\boldsymbol{x}_n)_{n\ge 0}$ to be uniformly distributed modulo one (for $\boldsymbol{h} = \mathbf{0}$ we trivially have equality for any sequence). This fact is the famous Weyl criterion for uniform distribution modulo one.

Let us make this assertion a bit more plausible. We consider the one-dimensional case and we identify the unit interval $[0, 1)$ equipped with addition modulo one, i.e., $\mathbb{R}/\mathbb{Z}$, with the one-dimensional torus $(\mathbb{T}, \cdot)$, where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, via the group isomorphism $x \mapsto \mathrm{e}^{2\pi \mathrm{i} x}$. Let $(x_n)_{n\ge 0}$ be a sequence in $[0, 1)$. Then $\frac{1}{N} \sum_{n=0}^{N-1} \mathrm{e}^{2\pi \mathrm{i} x_n}$ is nothing else than the centroid of the N points $\mathrm{e}^{2\pi \mathrm{i} x_0}, \ldots, \mathrm{e}^{2\pi \mathrm{i} x_{N-1}} \in \mathbb{T}$. If the centroid is now close to the origin then the points are evenly balanced on the torus $\mathbb{T}$ whereas this is not the case when the points are badly balanced (see Figure 3.2). However, this need not mean that the sequence $(x_n)_{n\ge 0}$ is uniformly distributed modulo one. For example consider the case where $x_{2k} = 0$ and $x_{2k+1} = 1/2$ for all $k \in \mathbb{N}_0$. Then $(x_n)_{n\ge 0}$ is obviously not uniformly distributed but the centroid of the points $\mathrm{e}^{2\pi \mathrm{i} x_0}, \ldots, \mathrm{e}^{2\pi \mathrm{i} x_{N-1}}$ tends to the origin when N grows to infinity. Thus for uniform distribution one needs more than just the property that the centroid of the points, transformed onto the torus, is close to the origin. The Weyl criterion states that it is enough to demand this property for the sequence $(\{h x_n\})_{n\ge 0}$ for all integers $h \neq 0$.

Theorem 3.5 (Weyl criterion) *A sequence $\mathcal{S} = (\boldsymbol{x}_n)_{n\ge 0}$ in the s-dimensional unit cube $[0, 1)^s$ is uniformly distributed modulo one, if and only if*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{h}\cdot\boldsymbol{x}_n} = 0 \tag{3.5}$$

holds for all vectors $\boldsymbol{h} \in \mathbb{Z} \setminus \{\mathbf{0}\}$.

Proof The Weyl criterion follows from the criterion in Theorem 3.4 by using the fact that the trigonometric polynomials of the form $\sum_{|\boldsymbol{h}|_\infty \le R} a_{\boldsymbol{h}} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{h}\cdot\boldsymbol{x}}$ with complex coefficients $a_{\boldsymbol{h}}$ and arbitrarily large $R \in \mathbb{N}_0$, are dense with respect to the uniform norm in the space of all continuous, complex-valued functions on $[0, 1]^s$. A detailed proof for the case $s = 1$ can be found in the book of Kuipers & Niederreiter [128, p. 7,8, Theorem 2.1]. See also [61]. □

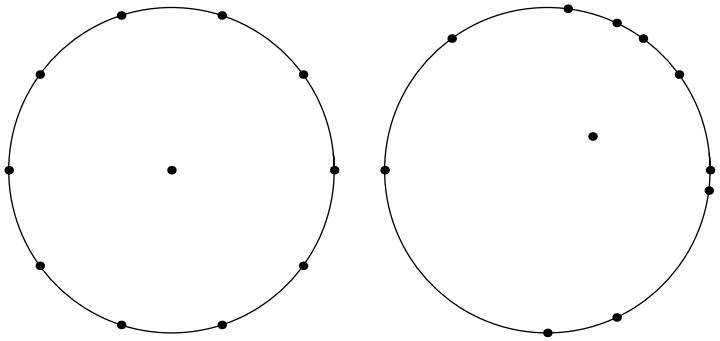

Figure 3.2 Ten points on the unit-circle. The points on the left are perfectly balanced and hence the centroid is exactly the origin. For the points on the right the centroid is far away from the origin.

Example 3.6 Applying the Weyl criterion to the sequence $(\{n\boldsymbol{\alpha}\})_{n\geq 0}$, where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_s) \in \mathbb{R}^s$ and where $\{\cdot\}$ denotes the fractional part applied component wise to a vector, we find that this sequence is uniformly distributed modulo one, if and only if the numbers $1, \alpha_1, \ldots, \alpha_s$ are linearly independent over $\mathbb{Q}$. Namely, if we assume that this holds true, then for each nonzero integer vector $\boldsymbol{h}$ we have $\boldsymbol{h} \cdot \boldsymbol{\alpha} \notin \mathbb{Z}$. Therefore, using the periodicity of the fractional part and the formula for a geometric sum, we have

$$\left| \sum_{n=0}^{N-1} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{h}\cdot\boldsymbol{x}_n} \right| = \left| \sum_{n=0}^{N-1} \mathrm{e}^{2\pi \mathrm{i} n \boldsymbol{h}\cdot\boldsymbol{\alpha}} \right| = \left| \frac{\mathrm{e}^{2\pi \mathrm{i} N \boldsymbol{h}\cdot\boldsymbol{\alpha}} - 1}{\mathrm{e}^{2\pi \mathrm{i} \boldsymbol{h}\cdot\boldsymbol{\alpha}} - 1} \right| \leq \frac{2}{|\mathrm{e}^{2\pi \mathrm{i} \boldsymbol{h}\cdot\boldsymbol{\alpha}} - 1|}$$

and hence by the Weyl criterion it follows that the sequence $(\{n\boldsymbol{\alpha}\})_{n\geq 0}$ is uniformly distributed modulo one.

If, on the other hand, $\boldsymbol{h}^* \cdot \boldsymbol{\alpha} \in \mathbb{Z}$ for some integer vector $\boldsymbol{h}^* \neq \boldsymbol{0}$, then $\frac{1}{N} \sum_{n=0}^{N-1} \mathrm{e}^{2\pi \mathrm{i} n \boldsymbol{h}^*\cdot\boldsymbol{\alpha}} = \frac{1}{N} \sum_{n=0}^{N-1} 1 = 1$, which implies that the Weyl criterion is not satisfied and the sequence $(\{n\boldsymbol{\alpha}\})_{n\geq 0}$ is not uniformly distributed.

As Walsh functions play a very important role in this book, we present also the Weyl criterion for the Walsh function system in more detail. See Appendix A for the definition and basic properties of Walsh functions.

Before we state the Weyl criterion for the Walsh function system, let us consider an example in dimension $s = 1$ and base $b = 2$. For $k \in \mathbb{N}_0$ with 2-adic expansion $k = \kappa_0 + \kappa_1 2 + \cdots + \kappa_r 2^r$ the 2-adic Walsh function is given by ${}_2\mathrm{wal}_k(x) = (-1)^{\xi_1\kappa_0 + \cdots + \xi_{r+1}\kappa_r}$ for $x \in [0,1)$ with canonical 2-adic expansion $x = \xi_1 2^{-1} + \xi_2 2^{-2} + \cdots$. This is of course a step function defined on the unit interval $[0,1)$; see Figure 3.3 for some examples. Hence, if a sequence $(x_n)_{n\geq 0}$ in $[0,1)$ is uniformly distributed modulo one, then

we have $\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} {}_2\mathrm{wal}_k(x_n) = \int_0^1 {}_2\mathrm{wal}_k(x)\,\mathrm{d}x$. This can be seen in the following way: if $k = 0$ then ${}_2\mathrm{wal}_k(x) = 1$ and $\int_0^1 {}_2\mathrm{wal}_k(x)\,\mathrm{d}x = 1$ and the equality holds trivially. If $k > 0$, then the last integral is zero, the Walsh function is 1 on a union of intervals with combined length $1/2$ and -1 on a union of intervals with combined length $1/2$. As $(x_n)_{n\geq 0}$ is uniformly distributed it follows that the equality also holds for $k > 0$.

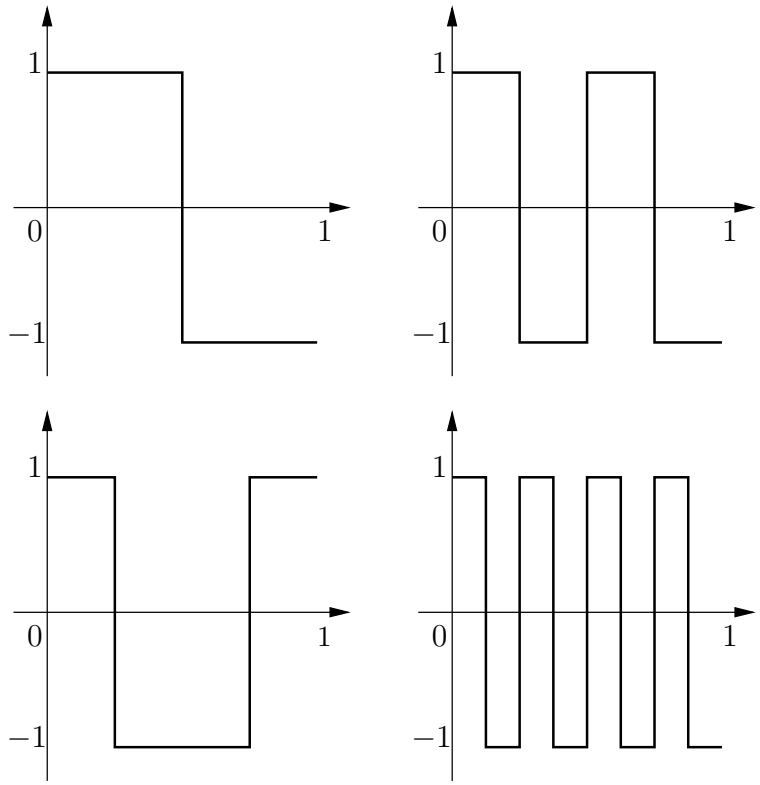

Figure 3.3 The 2-adic Walsh functions ${}_2\mathrm{wal}_1(x)$, ${}_2\mathrm{wal}_2(x)$, ${}_2\mathrm{wal}_3(x)$ and ${}_2\mathrm{wal}_4(x)$.

On the other hand, assume that for a given sequence $(x_n)_{n\geq 0}$ in $[0,1)$ we have $\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} {}_2\mathrm{wal}_1(x_n) = 0 = \int_0^1 {}_2\mathrm{wal}_1(x)\,\mathrm{d}x$. The first Walsh function ${}_2\mathrm{wal}_1(x)$ is constant with value 1 on the interval $[0,1/2)$ and constant with value -1 on the interval $[1/2,1)$; see Figure 3.3. Hence, asymptotically, in each of these two intervals we must have the same proportion of points of the sequence, namely $1/2$. Assume further that we also have $\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} {}_2\mathrm{wal}_k(x_n) = 0$ for $k \in \{2,3\}$. The Walsh functions ${}_2\mathrm{wal}_2(x)$ and ${}_2\mathrm{wal}_3(x)$ are both constant on the intervals $J_1 = [0,1/4)$,

$J_2 = [1/4, 1/2)$, $J_3 = [1/2, 3/4)$ and $J_4 = [3/4, 1)$ of length $1/4$; see again Figure 3.3. Let j_l be the proportion of points of the sequence $(x_n)_{n\geq 0}$ that belong to the interval J_l, $l \in \{1, 2, 3, 4\}$. Then we have from the above that $j_1 + j_2 = 1/2$ and $j_3 + j_4 = 1/2$, and from the asymptotic relation for the third and fourth Walsh function we obtain (see Figure 3.3) that $j_1 - j_2 + j_3 - j_4 = 0$ and $j_1 - j_2 - j_3 + j_4 = 0$. From these four equations we easily find that $j_1 = j_2 = j_3 = j_4 = 1/4$. Hence each interval J_l, $l \in \{1, 2, 3, 4\}$, contains in the limit the same proportion of points of the sequence $(x_n)_{n\geq 0}$.

If $\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} {}_2\mathrm{wal}_k(x_n) = 0$ for all $0 \leq k < 2^r$, then, by an extension of the above argument, one can conjecture (and we see below that this does indeed hold) that each interval of the form $\left[\frac{a}{2^r}, \frac{a+1}{2^r}\right)$, with $a \in \{0, \ldots, 2^r - 1\}$, of length 2^{-r}, contains in the limit the same proportion of points of the sequence, namely 2^{-r}, and hence (3.2) holds for all of these intervals. As one can approximate any subinterval $[x, y) \subseteq [0, 1)$ arbitrary closely by intervals of the form $\left[\frac{a}{2^r}, \frac{a+1}{2^r}\right)$, it follows that (3.2) holds for any subinterval $[x, y) \subseteq [0, 1)$ which means that the sequence $(x_n)_{n\geq 0}$ is uniformly distributed modulo one.

Now let us state the general result together with a detailed proof.

Theorem 3.7 (Weyl criterion for the Walsh function system) *Let $b \geq 2$ be an integer. A sequence $\mathcal{S} = (\boldsymbol{x}_n)_{n\geq 0}$ in the s-dimensional unit cube $[0, 1)^s$ is uniformly distributed modulo one, if and only if*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} {}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_n) = 0 \tag{3.6}$$

holds for all vectors $\boldsymbol{k} \in \mathbb{N}_0^s \setminus \{\boldsymbol{0}\}$.

From the point of view of Definition 3.1, the Weyl criterion for the Walsh function system seems to be more natural than the classical Weyl criterion using trigonometric functions, since by using the Walsh function system one measures directly the proportion of points of a sequence in certain intervals. However, if we identify the unit-interval $[0, 1)$ with the torus, and therefore define uniform distribution on the torus, then the classical Weyl criterion using trigonometric functions becomes more natural.

For the proof of the result we need some preparation. The following fundamental definition is used throughout the book.

Definition 3.8 Let $b \geq 2$ be an integer. An s-dimensional, b-adic *elemen-*

tary interval is an interval of the form

$$\prod_{i=1}^{s} \left[\frac{a_i}{b^{d_i}}, \frac{a_i+1}{b^{d_i}} \right)$$

with integers $0 \le a_i < b^{d_i}$ and $d_i \ge 0$ for all $1 \le i \le s$. If $d_1, \ldots, d_s$ are such that $d_1 + \cdots + d_s = k$, then we say that the elementary interval is of *order* k.

Lemma 3.9 *For $b \ge 2$ consider an s-dimensional, b-adic elementary interval $J = \prod_{i=1}^{s} \left[\frac{a_i}{b^{d_i}}, \frac{a_i+1}{b^{d_i}} \right)$ with integers $0 \le a_i < b^{d_i}$ and $d_i \in \mathbb{N}_0$ for all $1 \le i \le s$. Let further $\boldsymbol{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s \setminus \{\boldsymbol{0}\}$ be such that $k_i \ge b^{d_i}$ for at least one index $1 \le i \le s$. Then for the $\boldsymbol{k}$th Walsh coefficient of the characteristic function of J we have*

$$\widehat{\chi}_J(\boldsymbol{k}) = 0.$$

Proof First we show the one-dimensional case. Let $k \ge b^d$ and $0 \le a < b^d$ be integers with b-adic expansions $k = \kappa_0 + \kappa_1 b + \cdots + \kappa_g b^g$ with $\kappa_g \ne 0$ and $g \ge d$, and $a = \alpha_0 + \alpha_1 b + \cdots + \alpha_{d-1} b^{d-1}$. For $x \in \left[\frac{a}{b^d}, \frac{a+1}{b^d} \right)$, we have that the b-adic expansion of x is of the form

$$x = \frac{\alpha_{d-1}}{b} + \frac{\alpha_{d-2}}{b^2} + \cdots + \frac{\alpha_0}{b^d} + \sum_{j \ge d+1} \frac{\xi_j}{b^j},$$

where $\xi_j \in \{0, \ldots, b-1\}$ are arbitrary b-adic digits for $j \ge d+1$. Therefore we obtain

$$\begin{aligned} \int_{a/b^d}^{(a+1)/b^d} {}_b\mathrm{wal}_k(x)\,\mathrm{d}x &= \omega_b^{\kappa_0\alpha_{d-1}+\cdots+\kappa_{d-1}\alpha_0} \int_{a/b^d}^{(a+1)/b^d} \omega_b^{\kappa_d\xi_{d+1}+\cdots+\kappa_g\xi_{g+1}}\,\mathrm{d}x \\ &= \omega_b^{\kappa_0\alpha_{d-1}+\cdots+\kappa_{d-1}\alpha_0} \frac{1}{b^{g+1}} \prod_{j=d}^{g} \sum_{\zeta=0}^{b-1} \omega_b^{\kappa_j\zeta} = 0, \end{aligned}$$

as for $\kappa \ne 0$ we have $\sum_{\zeta=0}^{b-1} \omega_b^{\kappa\zeta} = (\omega_b^{\kappa b} - 1)/(\omega_b - 1) = 0$.

Now let $\boldsymbol{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ with $k_i \ge b^{d_i}$ for at least one index $1 \le i \le s$. Then we obtain from the above that for $J = \prod_{i=1}^{s} \left[\frac{a_i}{b^{d_i}}, \frac{a_i+1}{b^{d_i}} \right)$ we have

$$\begin{aligned} \overline{\widehat{\chi}_J(\boldsymbol{k})} &= \int_{[0,1]^s} \chi_J(\boldsymbol{x})\,{}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_J {}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\ &= \prod_{i=1}^{s} \int_{a_i/b^{d_i}}^{(a_i+1)/b^{d_i}} {}_b\mathrm{wal}_{k_i}(x)\,\mathrm{d}x = 0. \end{aligned}$$

□

Proof of Theorem 3.7 Assume first that the sequence $\mathcal{S} = (\boldsymbol{x}_n)_{n\geq 0}$ is uniformly distributed modulo one. As the Walsh function ${}_b\mathrm{wal}_{\boldsymbol{k}}$ with $\boldsymbol{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$, $0 \leq k_i < b^{r_i}$ for all $1 \leq i \leq s$, is constant on each interval of the form

$$J = \prod_{i=1}^{s} \left[\frac{a_i}{b^{r_i}}, \frac{a_i + 1}{b^{r_i}} \right),$$

with integers $0 \leq a_i < b^{r_i}$ for all $1 \leq i \leq s$, we can write ${}_b\mathrm{wal}_{\boldsymbol{k}}$ as a step function, i.e.,

$${}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) = \sum_{l \in M} c_l \chi_{J_l}(\boldsymbol{x})$$

with a finite set M, $c_l \in \mathbb{R}$ and pairwise disjoint intervals $J_l \subseteq [0,1]^s$. Then we obtain

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} {}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_n) = \sum_{l \in M} c_l \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \chi_{J_l}(\boldsymbol{x}_n) = \sum_{l \in M} c_l \lambda_s(J_l),$$

as the sequence $(\boldsymbol{x}_n)_{n\geq 0}$ is uniformly distributed modulo one. Since for $\boldsymbol{k} \neq \boldsymbol{0}$ we have

$$0 = \int_{[0,1]^s} {}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = \sum_{l \in M} c_l \lambda_s(J_l)$$

we find that (3.6) holds for all $\boldsymbol{k} \neq \boldsymbol{0}$.

Now we show the other direction. Assume that (3.6) holds for all $\boldsymbol{k} \neq \boldsymbol{0}$. Let first J be a b-adic elementary interval with $d_1 = \cdots = d_s = r \in \mathbb{N}_0$, i.e., of the form

$$J = \prod_{i=1}^{s} \left[\frac{a_i}{b^r}, \frac{a_i + 1}{b^r} \right), \tag{3.7}$$

with integers $0 \leq a_i < b^r$ for all $1 \leq i \leq s$. Then it follows from Lemma 3.9 that the characteristic function of J has a finite Walsh series representation, i.e.,

$$\chi_J(\boldsymbol{x}) = \sum_{\substack{\boldsymbol{k} \in \mathbb{N}_0^s \\ |\boldsymbol{k}|_\infty < b^r}} \widehat{\chi}_J(\boldsymbol{k})\, {}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}),$$

where for $\boldsymbol{k} = (k_1, \ldots, k_s)$ we write $|\boldsymbol{k}|_\infty := \max_{1\leq j\leq s} |k_j|$. Hence

$$\lim_{N\to\infty} \frac{A(J, N, \mathcal{S})}{N} = \sum_{\substack{\boldsymbol{k} \in \mathbb{N}_0^s \\ |\boldsymbol{k}|_\infty < b^r}} \widehat{\chi}_J(\boldsymbol{k}) \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} {}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_n) = \widehat{\chi}_J(\boldsymbol{0}) = \lambda_s(J),$$

by our assumption. Let now $[\boldsymbol{x}, \boldsymbol{y}) \subseteq [0,1)^s$ be an arbitrary interval with $\boldsymbol{x} =$

$(x_1, \ldots, x_s)$ and $\boldsymbol{y} = (y_1, \ldots, y_s)$. For $r \in \mathbb{N}_0$ choose $u_i, v_i \in \{0, \ldots, b^r - 1\}$ such that $u_i \le x_i b^r < u_i + 1$ and $v_i \le y_i b^r < v_i + 1$ for $1 \le i \le s$. Then we have

$$J_1 := \prod_{i=1}^{s} \left[\frac{u_i + 1}{b^r}, \frac{v_i}{b^r} \right) \subseteq [\boldsymbol{x}, \boldsymbol{y}) \subseteq \prod_{i=1}^{s} \left[\frac{u_i}{b^r}, \frac{v_i + 1}{b^r} \right) =: J_2$$

and $A(J_1, N, \mathcal{S}) \le A([\boldsymbol{x}, \boldsymbol{y}), N, \mathcal{S}) \le A(J_2, N, \mathcal{S})$. As both, J_1 and J_2 are disjoint, finite unions of b-adic elementary intervals of the form (3.7), we obtain from the above considerations that

$$\prod_{i=1}^{s} \frac{v_i - u_i - 1}{b^r} \le \lim_{N \to \infty} \frac{A([\boldsymbol{x}, \boldsymbol{y}), N, \mathcal{S})}{N} \le \prod_{i=1}^{s} \frac{v_i - u_i + 1}{b^r}.$$

But, as for all $1 \le i \le s$ we have $\lim_{r \to \infty} \frac{v_i - u_i - 1}{b^r} = \lim_{r \to \infty} \frac{v_i - u_i + 1}{b^r} = y_i - x_i$, it follows that

$$\lim_{N \to \infty} \frac{A([\boldsymbol{x}, \boldsymbol{y}), N, \mathcal{S})}{N} = \lambda_s([\boldsymbol{x}, \boldsymbol{y})),$$

and hence the sequence $\mathcal{S}$ is uniformly distributed modulo one. □

Now we can easily give a further example for a uniformly distributed sequence.

Definition 3.10 Let $b \ge 2$ be an integer. For any $n \in \mathbb{N}_0$ with b-adic expansion $n = n_0 + n_1 b + n_2 b^2 + \cdots$ (this expansion is obviously finite) the (b-adic) *radical inverse function* $\varphi_b : \mathbb{N}_0 \to [0, 1)$ is defined as

$$\varphi_b(n) = \frac{n_0}{b} + \frac{n_1}{b^2} + \cdots .$$

Then the *b-adic van der Corput sequence* is defined as the one-dimensional sequence $\mathcal{S} = (x_n)_{n \ge 0}$ with $x_n = \varphi_b(n)$ for all $n \in \mathbb{N}_0$.

Example 3.11 For $b = 2$ the first elements of the 2-adic van der Corput sequence are $0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{5}{8}, \frac{3}{8}, \frac{7}{8}, \frac{1}{16}, \ldots$.

Example 3.12 Using Theorem 3.7 we show that the b-adic van der Corput sequence is uniformly distributed modulo one. Let $k \in \mathbb{N}$ with $k = \kappa_0 + \kappa_1 b + \cdots + \kappa_{r-1} b^{r-1}$, where $\kappa_{r-1} \ne 0$. For the b-adic van der Corput sequence the nth element is of the form $x_n = n_0 b^{-1} + n_1 b^{-2} + \cdots$ and hence we have

$$E(N) := \sum_{n=0}^{N-1} {}_b\mathrm{wal}_k(x_n) = \sum_{n=0}^{N-1} \omega_b^{\kappa_0 n_0 + \cdots + \kappa_{r-1} n_{r-1}}.$$

Let first $N = b^r$. Then we have $E(b^r) = \prod_{j=0}^{r-1} \sum_{n=0}^{b-1} \omega_b^{\kappa_j n} = 0$ as at least

$\kappa_{r-1} \neq 0$ and the same holds for each multiple of b^r, i.e., $E(vb^r) = 0$ for all $v \in \mathbb{N}$. From this we find that $|E(N)| \leq b^r$ for all $N \in \mathbb{N}$ and hence

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} {}_b\mathrm{wal}_k(x_n) = 0.$$

Now it follows from the Weyl criterion for the Walsh function system that the b-adic van der Corput sequence is uniformly distributed modulo one.

3.2 Discrepancy

In the last section we found that a sequence should be uniformly distributed modulo one to satisfy our purpose of approximating the integral of a Riemann integrable function arbitrarily closely with a QMC algorithm using the first N points of this sequence. In practice, however, we can only use finite sets of sample points (where here and throughout this book by a set of points we always mean a multi-set where the multiplicity of elements matters). But a finite sequence can never be uniformly distributed modulo one. Nevertheless, Theorem 3.3 and the results from Chapter 2 suggest to use point sets whose empirical distribution is close to uniform distribution modulo one.

In the following we introduce several quantitative measures for the deviation of a finite point set from uniform distribution. Some of them have already been introduced and used in Chapter 2. Such measures are usually called discrepancies. The definition of uniform distribution modulo one leads directly to the following definition.

Definition 3.13 Let $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$ be a finite point set in $[0,1)^s$. The *extreme discrepancy* D_N of this point set is defined as

$$D_N(\mathcal{P}) := \sup_J \left| \frac{A(J,N)}{N} - \lambda_s(J) \right|,$$

where the supremum is extended over all sub-intervals $J \subseteq [0,1)^s$ of the form $J = [\boldsymbol{a}, \boldsymbol{b})$. For an infinite sequence $\mathcal{S}$ the extreme discrepancy $D_N(\mathcal{S})$ is the extreme discrepancy of the first N elements of the sequence.

It can be shown (see Exercise 3.9) that a sequence $\mathcal{S}$ is uniformly distributed modulo one, if and only if $\lim_{N\to\infty} D_N(\mathcal{S}) = 0$. Hence, for uniformly distributed sequences, the extreme discrepancy goes to zero as N tends to infinity. However, this convergence to zero cannot be arbitrarily fast. Consider, for example, an interval of volume $\varepsilon > 0$ which contains exactly one point of the first N elements of the sequence $\mathcal{S}$. Then by choosing $\varepsilon > 0$

arbitrarily small, we find $D_N(\mathcal{S}) \geq 1/N$. This gives a first lower bound on the extreme discrepancy.

Very often one uses a slightly weaker version of the extreme discrepancy which is commonly known as the star discrepancy. Here the supremum in Definition 3.13 is only extended over all subintervals of the unit cube with one vertex anchored at the origin. The star discrepancy D_N^* of a finite point set $\mathcal{P}$ has been introduced in Definition 2.14 as the sup-norm of the discrepancy function $\Delta_{\mathcal{P}}(\boldsymbol{x}) := A([\mathbf{0}, \boldsymbol{x}), N, \mathcal{P})/N - \lambda_s([\mathbf{0}, \boldsymbol{x}))$ (see Definition 2.13), which can be considered as a local measure for the deviation from uniform distribution. That is, for a point set $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$ in $[0,1)^s$ the star discrepancy is given by

$$D_N^*(\mathcal{P}) := \sup_{\boldsymbol{x} \in [0,1]^s} |\Delta_{\mathcal{P}}(\boldsymbol{x})| \,.$$

For an infinite sequence $\mathcal{S}$ the star discrepancy $D_N^*(\mathcal{S})$ is the star discrepancy of the first N elements of the sequence.

From these definitions we immediately obtain the following relation between the extreme discrepancy and the star discrepancy.

Proposition 3.14 *For any point set $\mathcal{P}$ consisting of N points in $[0,1)^s$ we have*

$$D_N^*(\mathcal{P}) \leq D_N(\mathcal{P}) \leq 2^s D_N^*(\mathcal{P}).$$

Proof The left inequality is obvious. For the right inequality we mention that any subinterval of $[0,1]^s$ can be written as composition of at most 2^s subintervals of $[0,1]^s$ with one vertex anchored in the origin. For example, for $s = 2$ and $\boldsymbol{a} = (a_1, a_2)$ and $\boldsymbol{b} = [b_1, b_2)$ we have

$$[\boldsymbol{a}, \boldsymbol{b}) = ([0, b_1) \times [0, b_2) \setminus [0, a_1) \times [0, b_2)) \setminus ([0, b_1) \times [0, a_2) \setminus [0, a_1) \times [0, a_2)) \,.$$

From this composition the result immediately follows. □

On account of Proposition 3.14 we mainly deal with the simpler star discrepancy instead of the extreme discrepancy in the following. A further motivation for concentrating on the star discrepancy is its appearance in the Koksma-Hlawka inequality (Theorem 2.18). Results for the extreme discrepancy can be obtained from results for the star discrepancy together with Proposition 3.14.

Obviously this also holds in the other direction. From our results for the extreme discrepancy we find now that a sequence is uniformly distributed modulo one, if and only if its star discrepancy tends to zero. Furthermore we find the (weak) lower bound $D_N^*(\mathcal{S}) \geq 1/(2^s N)$ for the star discrepancy of any sequence $\mathcal{S}$ in $[0,1)^s$.

In the following we present three important statements on the star discrepancy which are often used (sometimes implicitly) within this book.

Sometimes, it is very useful to know that the star discrepancy is a continuous function on $[0,1)^{Ns}$. This is more or less the assertion of the following proposition.

Proposition 3.15 *Let $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$ be a point set in $[0,1)^s$ with star discrepancy $D_N^*(\mathcal{P})$. Let $\boldsymbol{x}_n := (x_{n,1}, \ldots, x_{n,s})$ for $0 \le n \le N-1$, and let $\delta_{n,i}$, $0 \le n \le N-1$, $1 \le i \le s$, be nonnegative reals with $\delta_{n,i} < \varepsilon$, such that $x_{n,i} + \delta_{n,i} < 1$ for all $0 \le n \le N-1$ and $1 \le i \le s$. Then for the star discrepancy $D_N^*(\widetilde{\mathcal{P}})$ of the shifted point set $\widetilde{\mathcal{P}} = \{\widetilde{\boldsymbol{x}}_0, \ldots, \widetilde{\boldsymbol{x}}_{N-1}\}$, with $\widetilde{x}_{n,i} := x_{n,i} + \delta_{n,i}$ for all $0 \le n \le N-1$ and $1 \le i \le s$, we have*

$$|D_N^*(\mathcal{P}) - D_N^*(\widetilde{\mathcal{P}})| \le \varepsilon s.$$

Proof For an arbitrary interval $B = \prod_{i=1}^{s}[0,\alpha_i) \subseteq [0,1)^s$ we define for $j \in \{0,1\}$ the intervals

$$\widetilde{B}_j := \prod_{i=1}^{s} \left[0, \widetilde{\alpha}_i^{(j)}\right)$$

with

$$\widetilde{\alpha}_i^{(j)} := \begin{cases} 0 & \text{if } \alpha_i + (-1)^j\varepsilon < 0, \\ 1 & \text{if } \alpha_i + (-1)^j\varepsilon > 1, \\ \alpha_i + (-1)^j\varepsilon & \text{otherwise.} \end{cases}$$

Then one can prove, by induction on the dimension s, that $|\lambda_s(B) - \lambda_s(\widetilde{B}_j)| \le \varepsilon s$ holds for $j \in \{0,1\}$.

Then we have $A(\widetilde{B}_1, N, \mathcal{P}) \le A(B, N, \widetilde{\mathcal{P}}) \le A(B, N, \mathcal{P})$ and hence

$$A(B, N, \widetilde{\mathcal{P}}) - N\lambda_s(B) \le |A(B, N, \mathcal{P}) - N\lambda_s(B)| \le D_N^*(\mathcal{P})$$

and

$$\begin{aligned} N\lambda_s(B) - A(B, N, \widetilde{\mathcal{P}}) &\le N\lambda_s(B) - A(\widetilde{B}_1, N, \mathcal{P}) \\ &= N\lambda_s(\widetilde{B}_1) - A(\widetilde{B}_1, N, \mathcal{P}) + N\lambda_s(B) - N\lambda_s(\widetilde{B}_1) \\ &\le D_N^*(\mathcal{P}) + N\varepsilon s. \end{aligned}$$

Therefore we have $|A(B, N, \widetilde{\mathcal{P}}) - N\lambda_s(B)| \le D_N^*(\mathcal{P}) + N\varepsilon s$. Since B is an arbitrary interval, we get from this inequality that $D_N^*(\widetilde{\mathcal{P}}) \le D_N^*(\mathcal{P}) + \varepsilon s$.

In the same way we can show that $D_N^*(\mathcal{P}) \le D_N^*(\widetilde{\mathcal{P}}) + \varepsilon s$, which shows the result. □

Sometimes it is possible to split a given point set into smaller point sets with low star discrepancies. In this case the following result, which is often

called the *triangle inequality for the discrepancy*, may be very useful to get an upper bound on the star discrepancy of the superposition of the small point sets (see [128, p. 115, Theorem 2.6]).

Proposition 3.16 *For $1 \le i \le k$, let $\mathcal{P}_i$ be point sets consisting of N_i points in $[0,1)^s$ with star discrepancy $D^*_{N_i}(\mathcal{P}_i)$. Let $\mathcal{P}$ be the point set obtained by listing in some order the terms of $\mathcal{P}_i$, $1 \le i \le k$. We set $N = N_1 + \cdots + N_k$, which is the number of points of $\mathcal{P}$. Then we have*

$$D^*_N(\mathcal{P}) \le \sum_{i=1}^{k} \frac{N_i}{N} D^*_{N_i}(\mathcal{P}_i),$$

and the same result holds with the star discrepancy replaced by the extreme discrepancy.

The third statement on the star discrepancy gives the error when we replace the supremum in its definition by a maximum over a finite, equidistant grid with given mesh-size. For an integer $m \ge 2$ let $\Gamma_m := \frac{1}{m}\mathbb{Z}^s \pmod 1$ be the equidistant grid with mesh-size $1/m$.

Proposition 3.17 *Let $\delta > 0$ and define $m = \lceil s/\delta \rceil$. Further let Γ_m be the equidistant grid on $[0,1]^s$ with mesh-size $1/m$. Then for any point set $\mathcal{P}$ consisting of N points in $[0,1)^s$ we have*

$$D^*_N(\mathcal{P}) \le \max_{\boldsymbol{y} \in \Gamma_m} |\Delta_{\mathcal{P}}(\boldsymbol{y})| + \delta.$$

For the proof of this result we need the following lemma.

Lemma 3.18 *Let $u_i, v_i \in [0,1]$ for $1 \le i \le s$ and let $\delta \in [0,1]$ be such that $|u_i - v_i| \le \delta$ for $1 \le i \le s$. Then*

$$\left| \prod_{i=1}^{s} u_i - \prod_{i=1}^{s} v_i \right| \le 1 - (1-\delta)^s \le s\delta.$$

Proof As in [175] we prove the result by induction on s. Trivially, the result holds for $s = 1$. Let $s > 1$. We may assume that $u_s \ge v_s$. Then, by assuming that the result holds true for $s-1$, we have

$$\begin{aligned}
\left| \prod_{i=1}^{s} u_i - \prod_{i=1}^{s} v_i \right| &= \left| (u_s - v_s) \prod_{i=1}^{s-1} u_i + v_s \left(\prod_{i=1}^{s-1} u_i - \prod_{i=1}^{s-1} v_i \right) \right| \\
&\le |u_s - v_s| + v_s(1 - (1-\delta)^{s-1}) \\
&= u_s - v_s(1-\delta)^{s-1} \\
&= u_s(1 - (1-\delta)^{s-1}) + (u_s - v_s)(1-\delta)^{s-1}
\end{aligned}$$

$$\begin{aligned}&\le 1-(1-\delta)^{s-1}+\delta(1-\delta)^{s-1}\\&= 1-(1-\delta)^s.\end{aligned}$$

This shows the first inequality. For the second inequality we consider the real function $x \mapsto x^s$. According to the Mean Value Theorem, for all reals $y > z$ we have $y^s - z^s = s\,\xi^{s-1}(y-z)$ for some $\xi \in (z,y)$. Now we obtain the result by using this insight with the choice $y = 1$ and $z = 1-\delta$. □

Proof of Proposition 3.17 For $\varepsilon > 0$ we choose $\boldsymbol{y}^* = (y_1^*, \ldots, y_s^*) \in [0,1)^s$ such that

$$D_N^*(\mathcal{P}) \le \left| \frac{A([\mathbf{0},\boldsymbol{y}^*), N, \mathcal{P})}{N} - \lambda_s([\mathbf{0},\boldsymbol{y}^*)) \right| + \varepsilon.$$

Now choose $\boldsymbol{x} = (x_1, \ldots, x_s)$ and $\boldsymbol{y} = (y_1, \ldots, y_s)$ in Γ_m with $x_i \le y_i^* < x_i + \frac{1}{m} =: y_i$ for all $1 \le i \le s$. Using Lemma 3.18 we obtain

$$\prod_{i=1}^s y_i - \prod_{i=1}^s x_i \le 1 - \left(1 - \frac{1}{m}\right)^s \le \frac{s}{m} \le \delta.$$

Hence we get $-\delta + \prod_{i=1}^s y_i \le \prod_{i=1}^s y_i^* \le \prod_{i=1}^s x_i + \delta$ and therefore we have

$$\begin{aligned}\frac{A([\mathbf{0},\boldsymbol{x}), N, \mathcal{P})}{N} - \lambda_s([\mathbf{0},\boldsymbol{x})) - \delta &\le \frac{A([\mathbf{0},\boldsymbol{y}^*), N, \mathcal{P})}{N} - \lambda_s([\mathbf{0},\boldsymbol{y}^*))\\ &\le \frac{A([\mathbf{0},\boldsymbol{y}), N, \mathcal{P})}{N} - \lambda_s([\mathbf{0},\boldsymbol{y})) + \delta.\end{aligned}$$

From these inequalities we get

$$D_N^*(\mathcal{P}) \le \max_{\boldsymbol{y}\in\Gamma_m} \left| \frac{A([\mathbf{0},\boldsymbol{y}), N, \mathcal{P})}{N} - \lambda_s([\mathbf{0},\boldsymbol{y})) \right| + \delta + \varepsilon.$$

Since $\varepsilon > 0$ can be chosen arbitrarily small, the result follows. □

Similarly, as the star discrepancy is defined as the supremum norm of the discrepancy function we may now introduce other notions of discrepancies by taking different norms of the discrepancy function. In particular, we take the L_q norm in the following (the special case of L_2-discrepancy has been introduced in Definition 2.14 already).

Definition 3.19 Let $1 \le q < \infty$ be a real number. For a point set $\mathcal{P}$ in $[0,1)^s$ the L_q-discrepancy is defined as the L_q norm of the discrepancy function, i.e.,

$$L_{q,N}(\mathcal{P}) := \left(\int_{[0,1]^s} |\Delta_{\mathcal{P}}(\boldsymbol{x})|^q \, \mathrm{d}\boldsymbol{x} \right)^{1/q}.$$

For an infinite sequence $\mathcal{S}$ the L_q-discrepancy $L_{q,N}(\mathcal{S})$ is the L_q-discrepancy of the first N elements of the sequence.

Obviously, for any point set $\mathcal{P}$ and any $1 \leq q < \infty$ we have $L_{q,N}(\mathcal{P}) \leq D_N^*(\mathcal{P})$. Conversely, it is also possible to give an upper bound for the star discrepancy in terms of L_q-discrepancy, see, for example, [61, Theorem 1.8]. From this it follows that a sequence is uniformly distributed modulo one, if and only if its L_q-discrepancy tends to zero.

We gave already some (trivial) lower bounds for the extreme and the star discrepancy of finite point sets in the unit cube. While it can be shown that these bounds are best possible in the order of magnitude in N for one-dimensional point sets (compare with Exercise 2.1) this is by no means true in higher dimension. The following remarkable result was first proved by Roth [226] in 1954.

Theorem 3.20 (Roth's lower bound on L_2-discrepancy) *For any dimension $s \in \mathbb{N}$ and for any point set $\mathcal{P}$ consisting of N points in the s-dimensional unit cube we have*

$$L_{2,N}(\mathcal{P}) \geq \frac{1}{N}\sqrt{\binom{\lfloor \log_2 N \rfloor + s + 1}{s-1}\frac{1}{2^{2s+4}}} \geq c_s \frac{(\log N)^{(s-1)/2}}{N},$$

where $c_s = \frac{1}{2^{2s+4}(\log 2)^{(s-1)/2}\sqrt{(s-1)!}}$.

Remark 3.21 The inequality in the above theorem also applies to the extreme- and the star discrepancy as $D_N(\mathcal{P}) \geq D_N^*(\mathcal{P}) \geq L_{2,N}(\mathcal{P})$.

The original proof of Roth's result can be found in [226] (this proof is in dimension $s = 2$ only, but can be easily generalised to the general case). A detailed proof (in arbitrary dimension) using the orthogonality relation of Rademacher functions can be found in the book of Kuipers & Niederreiter [128]. We also refer to the book of Beck & Chen [11].

The constant in Theorem 3.20 here is better than that in Kuipers & Niederreiter, see [128, p. 104]. This can be obtained by a few slight modifications in the proof as in [128], which we describe in the following.

For the proof of Theorem 3.20 we need several lemmas and some notation which we introduce in the following. Thereby we follow the proofs of [128, Chapter 2, Lemma 2.1–Lemma 2.5].

Let $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$ with $\boldsymbol{x}_n = (x_{n,1}, \ldots, x_{n,s})$ and let $\psi(x) = -{}_2\mathrm{wal}_1(x) = (-1)^{\xi_1+1}$ for $x \in \mathbb{R}$ with $x = X + \xi_1 2^{-1} + \xi_2 2^{-2} + \cdots$ where $X \in \mathbb{Z}$, $\xi_1, \xi_2, \ldots \in \{0, 1\}$ and infinitely many of the $\xi_1, \xi_2, \ldots$ are 0. Further, let t be a natural number such that $2^{t-1} > N$, which will be specified below. For a vector $\boldsymbol{r} = (r_1, \ldots, r_s) \in \mathbb{N}_0^s$ we define $|\boldsymbol{r}|_1 = r_1 + \cdots + r_s$.

For a vector $\boldsymbol{r} \in \mathbb{N}_0^s$ with $|\boldsymbol{r}|_1 = t-1$ we define a function $G_{\boldsymbol{r}}$ on $\mathbb{R}^s$ as follows: if there exists an $0 \le n < N$ such that

$$(\lfloor 2^{r_1} x_{n,1} \rfloor, \ldots, \lfloor 2^{r_s} x_{n,s} \rfloor) = (\lfloor x_1 \rfloor, \ldots, \lfloor x_s \rfloor),$$

then we set $G_{\boldsymbol{r}}(x_1, \ldots, x_s) = 0$. Otherwise we set

$$G_{\boldsymbol{r}}(x_1, \ldots, x_s) = \psi(x_1) \cdots \psi(x_s).$$

Furthermore we define

$$F_{\boldsymbol{r}}(x_1, \ldots, x_s) = G_{\boldsymbol{r}}(2^{r_1} x_1, \ldots, 2^{r_s} x_s)$$

and

$$F(x_1, \ldots, x_s) = \sum_{\substack{\boldsymbol{r} \in \mathbb{N}_0^s \\ |\boldsymbol{r}|_1 = t-1}} F_{\boldsymbol{r}}(x_1, \ldots, x_s).$$

Lemma 3.22 *For a given vector $\boldsymbol{r} \in \mathbb{N}_0^s$ with $|\boldsymbol{r}|_1 = t-1$ and for some i with $1 \le i \le s$ let $a = h2^{-r_i}$ and $b = m2^{-r_i}$, where $h, m \in \mathbb{Z}$ and $h < m$. Then, for any fixed $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_s$, we have*

$$\int_a^b F_{\boldsymbol{r}}(x_1, \ldots, x_s) \, \mathrm{d}x_i = 0.$$

Proof Using the substitution $t = 2^{r_i} x_i$ we have

$$\begin{aligned} \int_a^b F_{\boldsymbol{r}}(x_1, \ldots, x_s) \, \mathrm{d}x_i &= \int_a^b G_{\boldsymbol{r}}(2^{r_1} x_1, \ldots, 2^{r_s} x_s) \, \mathrm{d}x_i \\ &= \frac{1}{2^{r_i}} \int_h^m G_{\boldsymbol{r}}(2^{r_1} x_1, \ldots, t, \ldots, 2^{r_s} x_s) \, \mathrm{d}t. \end{aligned}$$

Split up the interval $[h, m]$ into subintervals of the form $[c, c+1]$ with integers c. Then the integrand $G_{\boldsymbol{r}}(2^{r_1} x_1, \ldots, t, \ldots, 2^{r_s} x_s)$ is zero on certain subintervals of these intervals. On the remaining intervals the integrand is equal to $\psi(2^{r_1} x_1) \cdots \psi(t) \cdots \psi(2^{r_s} x_s)$. But for any c we have

$$\int_c^{c+1} \psi(t) \, \mathrm{d}t = -\int_0^1 {}_2\mathrm{wal}_1(t) \, \mathrm{d}t = 0$$

and hence the result follows. □

Lemma 3.23 *We have*

$$\int_0^1 \cdots \int_0^1 x_1 \cdots x_s F(x_1, \ldots, x_s) \, \mathrm{d}x_1 \cdots \mathrm{d}x_s \ge \binom{t-1+s-1}{s-1} \frac{2^{t-1} - N}{2^{2(s+t-1)}}.$$

Proof First we show that for all $\boldsymbol{r} \in \mathbb{N}_0^s$ with $|\boldsymbol{r}|_1 = t-1$ we have

$$\int_0^1 \cdots \int_0^1 x_1 \cdots x_s F_{\boldsymbol{r}}(x_1, \ldots, x_s)\, \mathrm{d}x_1 \cdots \mathrm{d}x_s \ge \frac{2^{t-1} - N}{2^{2(s+t-1)}}.$$

Using the substitution $t_i = 2^{r_i} x_i$ for $1 \le i \le s$ we have

$$\begin{aligned}&\int_0^1 \cdots \int_0^1 x_1 \cdots x_s F_{\boldsymbol{r}}(x_1, \ldots, x_s)\, \mathrm{d}x_1 \cdots \mathrm{d}x_s \\ &= \frac{1}{2^{2|\boldsymbol{r}|_1}} \int_0^{2^{r_1}} \cdots \int_0^{2^{r_s}} t_1 \cdots t_s G_{\boldsymbol{r}}(t_1, \ldots, t_s)\, \mathrm{d}t_1 \cdots \mathrm{d}t_s\end{aligned}$$

We have

$$\int_{h_1}^{h_1+1} \cdots \int_{h_s}^{h_s+1} t_1 \cdots t_s G_{\boldsymbol{r}}(t_1, \ldots, t_s)\, \mathrm{d}t_1 \cdots \mathrm{d}t_s = 0$$

whenever $(h_1, \ldots, h_s) \in \mathbb{N}_0^s$ with

$$(h_1, \ldots, h_s) = (\lfloor 2^{r_1} x_{n,1} \rfloor, \ldots, \lfloor 2^{r_s} x_{n,s} \rfloor)$$

for some $0 \le n \le N-1$. Therefore we have

$$\begin{aligned}&\int_0^1 \cdots \int_0^1 x_1 \cdots x_s F_{\boldsymbol{r}}(x_1, \ldots, x_s)\, \mathrm{d}x_1 \cdots \mathrm{d}x_s \\ &= \frac{1}{2^{2(t-1)}} \sum_{\boldsymbol{h}}{}^{*} \int_{h_1}^{h_1+1} \cdots \int_{h_s}^{h_s+1} t_1 \cdots t_s G_{\boldsymbol{r}}(t_1, \ldots, t_s)\, \mathrm{d}t_1 \cdots \mathrm{d}t_s, \quad (3.8)\end{aligned}$$

where the sum $\sum_{\boldsymbol{h}}^{*}$ is over all lattice points $\boldsymbol{h} = (h_1, \ldots, h_s)$ with $0 \le h_i < 2^{r_i}$ for $1 \le i \le s$ and with $\boldsymbol{h} \ne (\lfloor 2^{r_1} x_{n,1} \rfloor, \ldots, \lfloor 2^{r_s} x_{n,s} \rfloor)$ for all $0 \le n \le N-1$. Hence this sum is extended over at least $2^{|\boldsymbol{r}|_1} - N = 2^{t-1} - N$ lattice points.

For any integer h we have

$$\int_h^{h+1} t\psi(t)\, \mathrm{d}t = -\int_h^{h+1/2} t\, \mathrm{d}t + \int_{h+1/2}^{h+1} t\, \mathrm{d}t = \frac{1}{4}$$

and hence

$$\sum_{\boldsymbol{h}}{}^{*} \int_{h_1}^{h_1+1} \cdots \int_{h_s}^{h_s+1} t_1 \cdots t_s G_{\boldsymbol{r}}(t_1, \ldots, t_s)\, \mathrm{d}t_1 \cdots \mathrm{d}t_s \ge \frac{2^{t-1} - N}{4^s}.$$

From this together with (3.8) it follows that

$$\int_0^1 \cdots \int_0^1 x_1 \cdots x_s F_{\boldsymbol{r}}(x_1, \ldots, x_s)\, \mathrm{d}x_1 \cdots \mathrm{d}x_s \ge \frac{2^{t-1} - N}{2^{2(s+t-1)}}.$$

To obtain the final result note that the number of vectors $\boldsymbol{r} \in \mathbb{N}_0^s$ with $|\boldsymbol{r}|_1 = t-1$ is given by $\binom{t-1+s-1}{s-1}$. □

Lemma 3.24 *We have*

$$\int_0^1 \cdots \int_0^1 F^2(x_1,\ldots,x_s)\,\mathrm{d}x_1\cdots\mathrm{d}x_s \le \binom{t-1+s-1}{s-1}.$$

Proof We have

$$\begin{aligned}
&\int_0^1 \cdots \int_0^1 F^2(x_1,\ldots,x_s)\,\mathrm{d}x_1\cdots\mathrm{d}x_s \\
&= \sum_{\substack{\boldsymbol{r}\in\mathbb{N}_0^s \\ |\boldsymbol{r}|_1=t-1}} \int_0^1 \cdots \int_0^1 F_{\boldsymbol{r}}^2(x_1,\ldots,x_s)\,\mathrm{d}x_1\cdots\mathrm{d}x_s \\
&\quad + \sum_{\substack{\boldsymbol{r},\boldsymbol{w}\in\mathbb{N}_0^s \\ \boldsymbol{r}\ne\boldsymbol{w} \\ |\boldsymbol{r}|_1=|\boldsymbol{w}|_1=t-1}} \int_0^1 \cdots \int_0^1 F_{\boldsymbol{r}}(x_1,\ldots,x_s)F_{\boldsymbol{w}}(x_1,\ldots,x_s)\,\mathrm{d}x_1\cdots\mathrm{d}x_s.
\end{aligned}$$

Now $F_{\boldsymbol{r}}^2(x_1,\ldots,x_s)\le 1$ and hence the first sum is bounded by $\binom{t-1+s-1}{s-1}$. It remains to show that the second sum is zero.

We show that each term in the second sum is zero. Choose $\boldsymbol{r},\boldsymbol{w}\in\mathbb{N}_0^s$, $\boldsymbol{r}\ne\boldsymbol{w}$, and $\boldsymbol{r}=(r_1,\ldots,r_s)$ and $\boldsymbol{w}=(w_1,\ldots,w_s)$. Then there exists an index $1\le i\le s$ such that $r_i\ne w_i$. Without loss of generality we may assume that $r_i<w_i$. For fixed $x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_s$ we show that

$$\int_0^1 F_{\boldsymbol{r}}(x_1,\ldots,x_s)F_{\boldsymbol{w}}(x_1,\ldots,x_s)\,\mathrm{d}x_i = 0.$$

The result then follows.

Using the substitution $t=2^{w_i}x_i$ transforms the above integral into

$$\frac{1}{2^{w_i}}\int_0^{2^{w_i}} G_{\boldsymbol{r}}(2^{r_1}x_1,\ldots,2^{r_i-w_i}t,\ldots,2^{r_s}x_s)G_{\boldsymbol{w}}(2^{w_1}x_1,\ldots,t,\ldots,2^{w_s}x_s)\,\mathrm{d}t.$$

Split the interval $[0,2^{w_i})$ into subintervals $[c,c+1)$with integers c. In such an interval the integrand is either identical to zero or equal to

$$\psi(2^{r_1}x_1)\cdots\psi(2^{r_i-w_i}t)\cdots\psi(2^{r_s}x_s)\psi(2^{w_1}x_1)\cdots\psi(t)\cdots\psi(2^{w_s}x_s).$$

Here the only dependence on t is in $\psi(2^{r_i-w_i}t)\psi(t)$ and hence it suffices to show that $\int_c^{c+1}\psi(2^{r_i-w_i}t)\psi(t)\,\mathrm{d}t$ is zero. Since $r_i-w_i<0$ it follows that $\psi(2^{r_i-w_i}t)=-{}_2\mathrm{wal}_1(2^{r_i-w_i}t)$ is constant on the interval $[c,c+1)$ and hence we have

$$\int_c^{c+1}\psi(2^{r_i-w_i}t)\psi(t)\,\mathrm{d}t = {}_2\mathrm{wal}_1(2^{r_i-w_i}c)\int_0^1 {}_2\mathrm{wal}_1(t)\,\mathrm{d}t = 0. \qquad \square$$

Lemma 3.25 *For $0 \le n < N$, we have*

$$\int_{x_{n,1}}^{1} \cdots \int_{x_{n,s}}^{1} F(x_1, \ldots, x_s)\, \mathrm{d}x_1 \cdots \mathrm{d}x_s = 0.$$

Proof It suffices to show that

$$\int_{x_{n,1}}^{1} \cdots \int_{x_{n,s}}^{1} F_{\boldsymbol{r}}(x_1, \ldots, x_s)\, \mathrm{d}x_1 \cdots \mathrm{d}x_s = 0$$

for all $0 \le n \le N-1$ and all $\boldsymbol{r} \in \mathbb{N}_0^s$ with $|\boldsymbol{r}|_1 = t-1$. For fixed n and for $1 \le i \le s$, let a_i be the least integral multiple of 2^{-r_i} that is $\ge x_{n,i}$. Then we have $\int_{x_{n,1}}^{1} \cdots \int_{x_{n,s}}^{1} = \int_{x_{n,1}}^{a_1} \cdots \int_{x_{n,s}}^{a_s} +$(sum of integrals in which for at least one variable x_i, we integrate over an interval $[a_i, 1]$). The first integral on the right-hand side is zero, since for all $(x_1, \ldots, x_s)$ in the interval $\prod_{l=1}^{s}[x_{n,l}, a_l]$ we have

$$(\lfloor 2^{r_1} x_1 \rfloor, \ldots, \lfloor 2^{r_1} x_1 \rfloor) = (\lfloor 2^{r_1} x_{n,1} \rfloor, \ldots, \lfloor 2^{r_1} x_{n,1} \rfloor)$$

and hence by definition $F_{\boldsymbol{r}}(x_1, \ldots, x_s) = G_{\boldsymbol{r}}(2^{r_1} x_1, \ldots, 2^{r_s} x_s) = 0$.

The remaining integrals however are zero by Lemma 3.22 and hence the result follows. □

Proof of Theorem 3.20 For $0 \le n \le N-1$, let $J_n := \prod_{i=1}^{s}(x_{n,i}, 1]$ and let $\chi_{J_n}(\boldsymbol{x})$ be the characteristic function of this interval. Then $A([\mathbf{0}, \boldsymbol{x}), N, \mathcal{P}) = \sum_{n=0}^{N-1} \chi_{J_n}(\boldsymbol{x})$, where $\boldsymbol{x} = (x_1, \ldots, x_s) \in [0,1]^s$. Therefore,

$$\begin{aligned}\int_{[0,1]^s} A([\mathbf{0}, \boldsymbol{x}), N, \mathcal{P}) F(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} &= \sum_{n=0}^{N-1} \int_{[0,1]^s} \chi_{J_n}(\boldsymbol{x}) F(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \\ &= \sum_{n=0}^{N-1} \int_{J_n} F(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = 0\end{aligned}$$

by Lemma 3.25. Hence, using Lemma 3.23 we obtain

$$\begin{aligned}\int_{[0,1]^s} (-N\Delta_{\mathcal{P}}(\boldsymbol{x})) F(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} &= N \int_0^1 \cdots \int_0^1 x_1 \cdots x_s F(x_1, \ldots, x_s)\, \mathrm{d}x_1 \cdots \mathrm{d}x_s \\ &\ge N \binom{t-1+s-1}{s-1} \frac{2^{t-1} - N}{2^{2(s+t-1)}}.\end{aligned}$$

Then

$$\left(N \binom{t-1+s-1}{s-1} \frac{2^{t-1} - N}{2^{2(s+t-1)}} \right)^2$$

$$\leq \left(\int_{[0,1]^s} (-N\Delta_{\mathcal{P}}(\boldsymbol{x}))F(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \right)^2$$
$$\leq \left(\int_{[0,1]^s} (-N\Delta_{\mathcal{P}}(\boldsymbol{x}))^2\,\mathrm{d}\boldsymbol{x} \right) \left(\int_{[0,1]^s} F(\boldsymbol{x})^2\,\mathrm{d}\boldsymbol{x} \right)$$
$$\leq \left(\int_{[0,1]^s} (-N\Delta_{\mathcal{P}}(\boldsymbol{x}))^2\,\mathrm{d}\boldsymbol{x} \right) \binom{t-1+s-1}{s-1}$$

by Cauchy-Schwarz' inequality and by Lemma 3.24. Hence we obtain

$$\int_{[0,1]^s} (N\Delta_{\mathcal{P}}(\boldsymbol{x}))^2\,\mathrm{d}\boldsymbol{x} \geq N^2 \binom{t-1+s-1}{s-1} 2^{-4(s+t-1)} (2^{t-1}-N)^2.$$

Let t now be the unique integer for which $2N < 2^{t-1} \leq 4N$. Then

$$\int_{[0,1]^s} (N\Delta_{\mathcal{P}}(\boldsymbol{x}))^2\,\mathrm{d}\boldsymbol{x} \geq N^4 2^{-4(t-1)} \binom{t-1+s-1}{s-1} 2^{-4s}$$
$$\geq \binom{t-1+s-1}{s-1} \frac{1}{2^{4s+8}}.$$

Further, $1+\log_2 N < t-1 \leq 2+\log_2 N$, which implies $2+\lfloor \log_2 N \rfloor = t-1$ and substituting on the right hand side above yields

$$\int_{[0,1]^s} (N\Delta_{\mathcal{P}}(\boldsymbol{x}))^2\,\mathrm{d}\boldsymbol{x} \geq \binom{\lfloor \log_2 N \rfloor + s + 1}{s-1} \frac{1}{2^{4s+8}}$$
$$\geq \frac{(\log N)^{s-1}}{(\log 2)^{s-1}(s-1)!\,2^{4s+8}},$$

where we used $\binom{\lfloor \log_2 N \rfloor + s + 1}{s-1} \geq \frac{(\lfloor \log_2 N \rfloor + 3)^{s-1}}{(s-1)!} \geq \frac{(\log_2 N)^{s-1}}{(s-1)!}$ and $\log_2 N = \frac{\log N}{\log 2}$ in the last inequality. Thus the result follows. □

As the L_q norm is monotone increasing in q it follows that Roth's lower bound holds for all L_q-discrepancies with $q \geq 2$, too. Furthermore, it was shown by Schmidt [236] that the same is true for all $1 < q < 2$. Summing up, for any $1 < q < \infty$ and any dimension s there exists a $c_{s,q} > 0$ with the following property: for any point set $\mathcal{P}$ consisting of N points in the s-dimensional unit cube we have

$$L_{q,N}(\mathcal{P}) \geq c_{s,q} \frac{(\log N)^{(s-1)/2}}{N}.$$

On the other hand, it is known that this bound is best possible in the order of magnitude in N as was shown first for the L_2-discrepancy by Davenport [29] for $s = 2$ and by Roth [227, 228] and Frolov [82] for arbitrary

dimensions $s \in \mathbb{N}$ and by Chen [21] for the general L_q case. But we know even more. For any $q > 1$, any dimension $s \in \mathbb{N}$ and any integer $N \geq 2$ there is an explicit construction of a point set $\mathcal{P}$ consisting of N points in the s-dimensional unit cube such that

$$L_{q,N}(\mathcal{P}) \leq C_{s,q} \frac{(\log N)^{(s-1)/2}}{N}.$$

Such a construction was first given by Davenport for $q = s = 2$ and by Chen & Skriganov [22] for the case $q = 2$ and arbitrary dimension s. Later Skriganov [242] generalised this construction to the L_q case with arbitrary $q > 1$. We are concerned with this topic in Chapter 16.

For the star discrepancy the situation is quite different. For $s = 2$ we have an improvement due to Schmidt [235] (see also [128]) or Béjian [12] who showed that there is a $c > 0$ (for example $c = 0.06$ as shown in [12]) such that for the star discrepancy of any point set $\mathcal{P}$ consisting of N points in the two-dimensional unit square we have

$$D_N^*(\mathcal{P}) \geq c \frac{\log N}{N}.$$

In dimension $s = 3$ it was shown by Beck [10] that for any $\varepsilon > 0$ there exists an $\overline{N}(\varepsilon)$ such that for any point set $\mathcal{P}$ consisting of $N \geq \overline{N}(\varepsilon)$ points in the three-dimensional unit cube we have

$$D_N^*(\mathcal{P}) \geq \frac{\log N (\log \log N)^{1/8-\varepsilon}}{N}.$$

An improvement of Becks result in dimension $s = 3$ has been shown by Bilyk & Lacey [15]. They showed that there is a choice of $0 < \eta < 1/2$ such that for any point set $\mathcal{P} \subset [0,1)^3$ of cardinality N we have

$$D_N^*(\mathcal{P}) \geq c \frac{(\log N)^{1+\eta}}{N}$$

for some constant $c > 0$. This breakthrough led then to the paper of Bilyk, Lacey & Vagharshakyan [16] where it is shown that for any $s \in \mathbb{N}$, $s \geq 2$, there is a $c_s > 0$ and a $0 < \eta_s < \frac{1}{2}$ with the property that for any point set $\mathcal{P}$ consisting of N points in the s-dimensional unit cube we have

$$D_N^*(\mathcal{P}) \geq c_s \frac{(\log N)^{(s-1)/2+\eta_s}}{N}.$$

This is the best result for dimensions $s \geq 3$ currently known.

If we consider infinite sequences, then it follows from Roth's lower bound that there exists a $c_s > 0$ such that for the star discrepancy of any sequence

$\mathcal{S}$ in the s-dimensional unit cube we have

$$D_N^*(\mathcal{S}) \geq c_s \frac{(\log N)^{s/2}}{N}$$

for infinitely many values of $N \in \mathbb{N}$. For a proof see, for example, [128, Chapter 2, Theorem 2.2].

However, the exact lower order of the star discrepancy in N is still one of the most famous open problems in the theory of uniform distribution modulo one. It is widely believed that there exists some $c_s > 0$ such that for any point set $\mathcal{P}$ consisting of N points in the s-dimensional unit cube $[0,1)^s$ the inequality

$$D_N^*(\mathcal{P}) > c_s \frac{(\log N)^{s-1}}{N}$$

holds true. This lower bound would be best possible for the star discrepancy D_N^* as we see later. For infinite sequences a lower bound for the star discrepancy of order $(\log N)^s/N$ for infinitely many values of $N \in \mathbb{N}$ is conjectured.

3.3 General bounds for the discrepancy

From the Weyl criterion (Theorem 3.5) we know that the behaviour of exponential sums is closely related to uniform distribution modulo one. The following important result, which is usually attributed to Erdős, Turán and Koksma, gives a quantitative version of this insight.

Theorem 3.26 (Erdős-Turán-Koksma inequality) *For the discrepancy of any point set $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$ in $[0,1)^s$ we have*

$$D_N(\mathcal{P}) \leq \left(\frac{3}{2}\right)^s \left(\frac{2}{m+1} + \sum_{\substack{\boldsymbol{h} \in \mathbb{Z}^s \\ 0 < |\boldsymbol{h}|_\infty \leq m}} \frac{1}{r(\boldsymbol{h})} \left| \frac{1}{N} \sum_{n=0}^{N-1} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{h} \cdot \boldsymbol{x}_n} \right| \right),$$

where m is an arbitrary positive integer and where $r(\boldsymbol{h}) = \prod_{i=1}^s \max(1, |h_i|)$ for $\boldsymbol{h} = (h_1, \ldots, h_s) \in \mathbb{Z}^s$.

A proof of this bound can be found in [61, Section 1.2.2]. See also [128, Chapter 2, Section 2] for the special case of $s = 1$.

In practice one is mainly concerned with point sets whose elements only have rational components. For such point sets Niederreiter [175, Theorem 3.10] proved a general upper bound for the discrepancy in terms of exponential sums. To formulate this result we need some notation.

For an integer $M \geq 2$, let $C(M) = (-M/2, M/2] \cap \mathbb{Z}$ and let $C_s(M)$

be the Cartesian product of s copies of $C(M)$. Furthermore, let $C_s^*(M) = C_s(M) \setminus \{\mathbf{0}\}$. For $h \in C(M)$ put

$$r(h,M) = \begin{cases} M \sin(\pi |h|/M) & \text{if } h \neq 0, \\ 1 & \text{if } h = 0. \end{cases}$$

For $\boldsymbol{h} = (h_1, \ldots, h_s) \in C_s(M)$, put $r(\boldsymbol{h}, M) = \prod_{i=1}^{s} r(h_i, M)$.

Theorem 3.27 *Let $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$ be a point set in the s-dimensional unit cube where $\boldsymbol{x}_n$ is of the form $\boldsymbol{x}_n = \{\boldsymbol{y}_n/M\}$ with $\boldsymbol{y}_n \in \mathbb{Z}^s$ for all $0 \leq n < N$, and let $M \geq 2$ be an integer. Then we have*

$$D_N(\mathcal{P}) \leq 1 - \left(1 - \frac{1}{M}\right)^s + \sum_{\boldsymbol{h} \in C_s^*(M)} \frac{1}{r(\boldsymbol{h}, M)} \left| \frac{1}{N} \sum_{n=0}^{N-1} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{h} \cdot \boldsymbol{y}_n / M} \right|.$$

For a proof of this theorem we refer to [175, Chapter 3].

In the following we consider point sets for which all coordinates of all points have a finite digit expansion in a fixed base $b \geq 2$. A bound similar to that of Theorem 3.27 on the star discrepancy of such point sets was first given by Niederreiter [168, Satz 2] (see also [175, Theorem 3.12]). An approach to this result by means of Walsh functions was described by Hellekalek [94, Theorem 1]. To formulate the result of Hellekalek we again need some notation.

Let $b \geq 2$ be an integer. For a vector $\boldsymbol{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ we put $\rho_b(\boldsymbol{k}) := \prod_{i=1}^{s} \rho_b(k_i)$ where for $k \in \mathbb{N}_0$ we set

$$\rho_b(k) := \begin{cases} 1 & \text{if } k = 0, \\ \frac{1}{b^{r+1} \sin(\pi \kappa_r / b)} & \text{if } b^r \leq k < b^{r+1} \text{ where } r \in \mathbb{N}_0 \end{cases}$$

and where κ_r is the most significant digit in the b-adic expansion of k.

Theorem 3.28 *Let $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$ be a point set in the s-dimensional unit cube where $\boldsymbol{x}_n$ is of the form $\boldsymbol{x}_n = \{\boldsymbol{y}_n/b^m\}$ with $\boldsymbol{y}_n \in \mathbb{Z}^s$, and integers $m \geq 1$ and $b \geq 2$. Then we have*

$$D_N^*(\mathcal{P}) \leq 1 - \left(1 - \frac{1}{b^m}\right)^s + \sum_{\substack{\boldsymbol{k} \in \mathbb{N}_0^s \\ 0 < |\boldsymbol{k}|_\infty < b^m}} \rho_b(\boldsymbol{k}) \left| \frac{1}{N} \sum_{n=0}^{N-1} {}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_n) \right|.$$

The proof of this results is based on the following idea. We only consider elementary b-adic intervals since we know from Lemma 3.9 that the characteristic functions of such intervals have a finite Walsh series representation. The remaining Walsh coefficients can be bounded independently of the chosen elementary interval. Then we approximate each interval with one corner

anchored in the origin by elementary b-adic intervals and in this way the result will follow.

For the complete proof of Theorem 3.28 we need the following lemma which provides the announced bound on the Walsh coefficients for the characteristic function of an interval.

Lemma 3.29 *Let $J = [0, \beta)$ with $0 < \beta < 1$, and let $b^r \le k < b^{r+1}$, where $r \in \mathbb{N}_0$. Then for the kth Walsh coefficient of the characteristic function of J we have*

$$|\widehat{\chi}_J(k)| \le \rho_b(k).$$

Proof Let $\beta = \beta_1 b^{-1} + \beta_2 b^{-2} + \cdots$ be the b-adic expansion of β and let $\beta(r) := \beta_1 b^{-1} + \cdots + \beta_r b^{-r}$. Then we can write the interval $[0, \beta(r))$ as a disjoint union of finitely many one-dimensional b-adic elementary intervals $J(a, b^r) = [ab^{-r}, (a+1)b^{-r})$ of order r. As $k \ge b^r$ it follows from Lemma 3.9 that for each of these intervals $J(a, b^r)$ we have $\int_{J(a,b^r)} {}_b\mathrm{wal}_k(x)\,\mathrm{d}x = 0$. Therefore we obtain

$$\widehat{\chi}_J(k) = \int_{\beta(r)}^{\beta} \overline{{}_b\mathrm{wal}_k(x)}\,\mathrm{d}x.$$

Let $k = \kappa_0 + \kappa_1 b + \cdots + \kappa_r b^r$ with $\kappa_r \ne 0$ and let $k(r) := k - \kappa_r b^r$. Then the Walsh function ${}_b\mathrm{wal}_{k(r)}$ is constant on the interval $[\beta(r), \beta(r) + b^{-r})$ with value ${}_b\mathrm{wal}_{k(r)}(\beta(r))$. Hence with $x = \xi_1 b^{-1} + \xi_2 b^{-2} + \cdots$ we obtain

$$\begin{aligned}\overline{\widehat{\chi}_J(k)} &= \int_{\beta(r)}^{\beta} {}_b\mathrm{wal}_k(x)\,\mathrm{d}x = \int_{\beta(r)}^{\beta} \omega_b^{\xi_1\kappa_0 + \cdots + \xi_r\kappa_{r-1} + \xi_{r+1}\kappa_r}\,\mathrm{d}x \\ &= {}_b\mathrm{wal}_{k(r)}(\beta(r)) \int_{\beta(r)}^{\beta} \omega_b^{\xi_{r+1}\kappa_r}\,\mathrm{d}x. \end{aligned} \tag{3.9}$$

For the last integral we split up the integration domain in β_{r+1} one-dimensional, b-adic elementary intervals of order $r+1$ and in a rest interval with length of at most $b^{-(r+1)}$. Then we obtain

$$\begin{aligned}\int_{\beta(r)}^{\beta} \omega_b^{\xi_{r+1}\kappa_r}\,\mathrm{d}x &= \sum_{l=0}^{\beta_{r+1}-1} \int_{\frac{\beta_1}{b} + \cdots + \frac{\beta_r}{b^r} + \frac{l}{b^{r+1}}}^{\frac{\beta_1}{b} + \cdots + \frac{\beta_r}{b^r} + \frac{l+1}{b^{r+1}}} \omega_b^{\xi_{r+1}\kappa_r}\,\mathrm{d}x + \int_{\frac{\beta_1}{b} + \cdots + \frac{\beta_{r+1}}{b^{r+1}}}^{\beta} \omega_b^{\xi_{r+1}\kappa_r}\,\mathrm{d}x \\ &= \frac{1}{b^{r+1}} \sum_{l=0}^{\beta_{r+1}-1} \omega_b^{l\kappa_r} + \omega_b^{\beta_{r+1}\kappa_r}(\beta - \beta(r+1)) \\ &= \frac{1}{b^{r+1}} \frac{\omega_b^{\beta_{r+1}\kappa_r} - 1}{\omega_b^{\kappa_r} - 1} + \omega_b^{\beta_{r+1}\kappa_r}(\beta - \beta(r+1)). \end{aligned} \tag{3.10}$$

From (3.9) and (3.10) we now obtain

$$\begin{aligned}|\widehat{\chi}_J(k)| &= \frac{1}{b^{r+1}} \left| \frac{\omega_b^{\beta_{r+1}\kappa_r} - 1}{\omega_b^{\kappa_r} - 1} + b^{r+1}\omega_b^{\beta_{r+1}\kappa_r}(\beta - \beta(r+1)) \right| \\ &= \frac{1}{b^{r+1}} \left| \omega_b^{\beta_{r+1}\kappa_r} \left(\frac{1}{\omega_b^{\kappa_r} - 1} + b^{r+1}(\beta - \beta(r+1)) \right) - \frac{1}{\omega_b^{\kappa_r} - 1} \right| \\ &\le \frac{1}{b^{r+1}} \left(\left| \frac{1}{\omega_b^{\kappa_r} - 1} + b^{r+1}(\beta - \beta(r+1)) \right| + \frac{1}{|\omega_b^{\kappa_r} - 1|} \right).\end{aligned}$$

For any $0 \le \gamma \le 1$ and any $z \in \mathbb{C}$, $z \ne 1$, with $|z| = 1$ we have $\left| \frac{1}{z-1} + \gamma \right| \le 1/|z-1|$. Applying this inequality to the term above we find that

$$|\widehat{\chi}_J(k)| \le \frac{1}{b^{r+1}} \frac{2}{|\omega_b^{\kappa_r} - 1|} = \frac{1}{b^{r+1}} \frac{1}{|\sin(\pi\kappa_r/b)|}.$$

Since $\kappa_r \in \{1, \ldots, b-1\}$ it follows that $0 < \pi\kappa_r/b < \pi$ and hence we can omit the absolute value for the sine function in the above term and the lemma is proved. □

Remark 3.30 In Lemma 14.8 below we provide the Walsh series expansion of the function $\widehat{\chi}_{[0,x)}(k)$.

Proof of Theorem 3.28 Let $\boldsymbol{x} = (x_1, \ldots, x_s) \in [0,1]^s$. For $1 \le i \le s$ define $a_i := \min\{a \in \{1, \ldots, b^m\} : x_i \le a \cdot b^{-m}\}$ and set $\boldsymbol{y} := \frac{1}{b^m}\boldsymbol{a}$ with $\boldsymbol{a} = (a_1, \ldots, a_s)$. Then we have

$$|\Delta_{\mathcal{P}}(\boldsymbol{x})| \le |\Delta_{\mathcal{P}}(\boldsymbol{x}) - \Delta_{\mathcal{P}}(\boldsymbol{y})| + |\Delta_{\mathcal{P}}(\boldsymbol{y})|.$$

As $\mathcal{P} \subseteq \frac{1}{b^m}\mathbb{Z}^s \pmod 1$ it follows that $A([\boldsymbol{0}, \boldsymbol{x}), N, \mathcal{P}) = A([\boldsymbol{0}, \boldsymbol{y}), N, \mathcal{P})$ and hence

$$|\Delta_{\mathcal{P}}(\boldsymbol{x})| \le |x_1 \cdots x_s - y_1 \cdots y_s| + |\Delta_{\mathcal{P}}(\boldsymbol{y})|. \tag{3.11}$$

Since $|x_i - y_i| \le \frac{1}{b^m}$ for all $1 \le i \le s$ we obtain with Lemma 3.18 that

$$|x_1 \cdots x_s - y_1 \cdots y_s| \le 1 - \left(1 - \frac{1}{b^m}\right)^s. \tag{3.12}$$

It remains to estimate $|\Delta_{\mathcal{P}}(\boldsymbol{y})|$. Obviously, the interval $G := [\boldsymbol{0}, \boldsymbol{y})$ can be written as a finite disjoint union of b-adic elementary intervals of the form $\prod_{i=1}^s \left[\frac{c_i}{b^m}, \frac{c_i+1}{b^m}\right)$ with integers $0 \le c_i < b^m$ for all $1 \le i \le s$. Hence it follows from Lemma 3.9 that $\widehat{\chi}_G(\boldsymbol{k}) = 0$ for all $\boldsymbol{k} \in \mathbb{N}_0^s$ with $|\boldsymbol{k}|_\infty \ge b^m$. Therefore,

and by invoking the identity $\widehat{\chi}_G(\mathbf{0}) = \lambda_s(G)$, we find

$$\Delta_{\mathcal{P}}(\boldsymbol{y}) = \frac{1}{N}\sum_{n=0}^{N-1}(\chi_G(\boldsymbol{x}_n) - \lambda_s(G)) = \sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^s \\ 0<|\boldsymbol{k}|_\infty<b^m}} \widehat{\chi}_G(\boldsymbol{k})\left(\frac{1}{N}\sum_{n=0}^{N-1} {}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_n)\right).$$

Using Lemma 3.29 it follows that $|\widehat{\chi}_G(\boldsymbol{k})| \le \rho_b(\boldsymbol{k})$ and hence we obtain

$$|\Delta_{\mathcal{P}}(\boldsymbol{y})| \le \sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^s \\ 0<|\boldsymbol{k}|_\infty<b^m}} \rho_b(\boldsymbol{k})\left|\frac{1}{N}\sum_{n=0}^{N-1} {}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_n)\right|. \tag{3.13}$$

From (3.11), (3.12) and (3.13) we now get

$$|\Delta_{\mathcal{P}}(\boldsymbol{x})| \le 1 - \left(1-\frac{1}{b^m}\right)^s + \sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^s \\ 0<|\boldsymbol{k}|_\infty<b^m}} \rho_b(\boldsymbol{k})\left|\frac{1}{N}\sum_{n=0}^{N-1} {}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_n)\right|.$$

As this bound holds for any $\boldsymbol{x} \in [0,1]^s$, it follows that the star discrepancy of $\mathcal{P}$ also satisfies this bound and hence the result is proved. □

Remark 3.31 Note that for the point set $\mathcal{P}$ as considered in Theorem 3.28 we also have

$$D_N^*(\mathcal{P}) \ge 1 - \left(1-\frac{1}{b^m}\right)^s.$$

This follows easily from the assumption that the components of the points of $\mathcal{P}$ are of the form a/b^m with $a \in \{0,\dots,b^m-1\}$ and hence $\mathcal{P} \subseteq [0,1-b^{-m}]^s$.

3.4 Discrepancy of special point sets and sequences

In this section we analyse the discrepancy of some classical constructions of point sets in $[0,1)^s$.

The regular lattice

If we think of a point set whose points are very uniformly distributed in the unit cube one immediately might have a regular lattice (or equidistant grid) in mind.

By a *regular lattice* of $N = m^s$ points in the s-dimensional unit cube we understand the point set

$$\Gamma_m := \frac{1}{m}\mathbb{Z}^s \pmod 1 = \left\{\left(\frac{n_1}{m},\dots,\frac{n_s}{m}\right) : 0 \le n_i < m \text{ for } 1 \le i \le s\right\}. \tag{3.14}$$

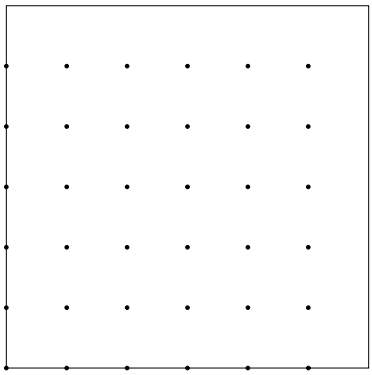

Figure 3.4 Regular lattice Γ_6 in $[0,1)^2$, i.e., $s=2$ and $m=6$.

However, we show in the next result that, with respect to discrepancy, this is not a good choice.

Proposition 3.32 *Let $m \geq 2$ be an integer. For the star discrepancy of the regular lattice Γ_m consisting of $N = m^s$ points in $[0,1)^s$ we have*

$$D_N^*(\Gamma_m) = 1 - \left(1 - \frac{1}{m}\right)^s.$$

Remark 3.33 Note that $N^{-1/s} \leq 1-(1-1/m)^s \leq s/m = sN^{-1/s}$, and hence for the star discrepancy of the regular lattice (3.14) we have

$$\frac{1}{N^{1/s}} \leq D_N^*(\Gamma_m) \leq \frac{s}{N^{1/s}}.$$

Proof of Proposition 3.32 As in Remark 3.31 we find that

$$D_N^*(\Gamma_m) \geq 1 - \left(1 - \frac{1}{m}\right)^s.$$

Now consider an arbitrary interval of the form $J = [0,\alpha_1) \times \cdots \times [0,\alpha_s)$. For $1 \leq i \leq s$ let $a_i \in \{0,\ldots,m-1\}$ be such that $a_i/m < \alpha_i \leq (a_i+1)/m$. Then we have $A(J,N,\Gamma_m) = \prod_{i=1}^s (a_i+1)$ and

$$0 \leq \frac{A(J,N,\Gamma_m)}{N} - \lambda_s(J) \leq \prod_{i=1}^s \frac{a_i+1}{m} - \prod_{i=1}^s \frac{a_i}{m}.$$

Therefore, and by invoking Lemma 3.18, we obtain

$$\left|\frac{A(J,N,\Gamma_m)}{N} - \lambda_s(J)\right| \leq \left|\prod_{i=1}^s \frac{a_i+1}{m} - \prod_{i=1}^s \frac{a_i}{m}\right| \leq 1 - \left(1-\frac{1}{m}\right)^s.$$

As J was chosen arbitrarily the result follows. (Alternatively, one may also use Theorem 3.27; see Exercise 3.21.) □

For dimension $s > 1$ the star discrepancy of the regular lattice is very poor. But for dimension $s = 1$ the order of the star discrepancy of the regular lattice is best possible.

For the centred regular lattice Γ_m^{c} consisting of $N = m^s$ points defined by (1.1) one can show in the same way as above (see Exercise 3.22) that

$$D_N^*(\Gamma_m^{\mathrm{c}}) = 1 - \left(1 - \frac{1}{2m}\right)^s.$$

Moreover, in dimension $s = 1$ it can be shown that the centred regular lattice $\Gamma_N^{\mathrm{c}} = \{x_n = \frac{2n+1}{2N} : 0 \le n < N\}$ has star discrepancy $D_N^*(\Gamma_N^{\mathrm{c}}) = \frac{1}{2N}$ (see Exercise 2.1). This is best possible among all point sets consisting of N points in $[0, 1)$. For a proof we refer to [128, Chapter 2, Corollary 1.2] or to [175, Theorem 2.6].

The van der Corput-Halton sequence

Now we turn to another construction which is a multidimensional generalisation of the van der Corput sequence as introduced in Definition 3.10.

Definition 3.34 Let $b_1, \ldots, b_s \ge 2$ be integers. The *van der Corput-Halton sequence* is the sequence $\mathcal{S} = (\boldsymbol{x}_n)_{n \ge 0}$ with $\boldsymbol{x}_n = (\varphi_{b_1}(n), \ldots, \varphi_{b_s}(n))$ for all $n \in \mathbb{N}_0$. Here φ_b is the b-adic radical inverse function as defined in Definition 3.10. The integers $b_1, \ldots, b_s$ are often called the *bases* of the van der Corput-Halton sequence.

Example 3.35 For dimension $s = 2$ and bases $b_1 = 2$ and $b_2 = 3$. The first points of the van der Corput-Halton sequence are given by $\boldsymbol{x}_0 = (0, 0)$, $\boldsymbol{x}_1 = (1/2, 1/3)$, $\boldsymbol{x}_2 = (1/4, 2/3)$, $\boldsymbol{x}_3 = (3/4, 1/9)$, $\boldsymbol{x}_4 = (1/8, 4/9)$, and so on. The first 1000 points of this sequence are shown in Figure 3.5.

It was known for a long time that, provided that the bases $b_1, \ldots, b_s$ are chosen to be pairwise relatively prime, the star discrepancy of the first N elements of the van der Corput-Halton sequence can be bounded by $c(b_1, \ldots, b_s)(\log N)^s/N + O((\log N)^{s-1}/N)$. For example, this was shown in [64, 87, 112, 161, 175]. Informally, one calls a sequence in the s-dimensional unit cube a *low-discrepancy sequence*, if its star discrepancy is of order $(\log N)^s/N$. While it is widely believed that this order of convergence is best possible for any infinite sequence in the s-dimensional unit cube, those results have a disadvantage for practical applications. Namely, the constant

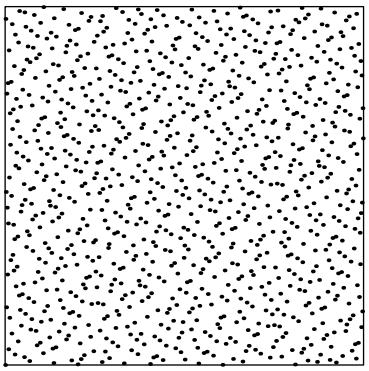

Figure 3.5 The first 1000 points of the two-dimensional van der Corput-Halton sequence in bases $b_1 = 2$ and $b_2 = 3$.

$c(b_1, \ldots, b_s) > 0$ depends very strongly on the dimension s. The minimal value for this quantity can be obtained if one chooses for $b_1, \ldots, b_s$ the first s prime numbers. But also in this case $c(b_1, \ldots, b_s)$ grows very fast to infinity if s increases.

This deficiency was remedied by Atanassov [6] who proved the following result.

Theorem 3.36 *Let $b_1, \ldots, b_s \geq 2$ be pairwise relatively prime integers and let $\mathcal{S}$ be the van der Corput-Halton sequence with bases $b_1, \ldots, b_s$. Then for any $N \geq 2$ we have*

$$ND_N^*(\mathcal{S}) \leq \frac{1}{s!} \prod_{i=1}^{s} \left(\frac{\lfloor b_i/2 \rfloor \log N}{\log b_i} + s \right) + \sum_{k=0}^{s-1} \frac{b_{k+1}}{k!} \prod_{i=1}^{k} \left(\frac{\lfloor b_i/2 \rfloor \log N}{\log b_i} + k \right).$$

In the following we present the proof of this result due to Atanassov [6]. From now on let $b_1, \ldots, b_s \geq 2$ be pairwise relatively prime integers and let $\mathcal{S}$ be the van der Corput-Halton sequence with bases $b_1, \ldots, b_s$.

Lemma 3.37 *Let J be an interval of the form $J = \prod_{i=1}^{s} [u_i/b_i^{m_i}, v_i/b_i^{m_i})$ with integers $0 \leq u_i < v_i < b_i^{m_i}$ and $m_i \geq 1$ for all $1 \leq i \leq s$. Then for the van der Corput-Halton sequence $\mathcal{S}$ the inequality*

$$|A(J, N, \mathcal{S}) - N\lambda_s(J)| \leq \prod_{i=1}^{s} (v_i - u_i)$$

holds for every $N \in \mathbb{N}$. Furthermore, for every $N \leq \prod_{i=1}^{s} b_i^{m_i}$ we have $A(J, N, \mathcal{S}) \leq \prod_{i=1}^{s} (v_i - u_i)$.

Proof For $n \in \mathbb{N}_0$ we denote the b_i-adic expansion by $n = n_0^{(i)} + n_1^{(i)} b_i + \cdots$. Choose $\boldsymbol{l} = (l_1, \ldots, l_s) \in \mathbb{N}_0^s$ with $0 \le l_i < b_i^{m_i}$ and with b_i-adic expansion $l_i = l_{i,m_i-1} + l_{i,m_i-2} b_i + \cdots + l_{i,0} b_i^{m_i-1}$ for all $1 \le i \le s$. We consider the interval

$$J_{\boldsymbol{l}} = \prod_{i=1}^{s} \left[\frac{l_i}{b_i^{m_i}}, \frac{l_i+1}{b_i^{m_i}} \right).$$

Then the nth element $\boldsymbol{x}_n$ of the van der Corput-Halton sequence is contained in $J_{\boldsymbol{l}}$, if and only if

$$\frac{l_{i,0}}{b_i} + \cdots + \frac{l_{i,m_i-1}}{b_i^{m_i}} \le \frac{n_0^{(i)}}{b_i} + \frac{n_1^{(i)}}{b_i^2} + \cdots < \frac{l_{i,0}}{b_i} + \cdots + \frac{l_{i,m_i-1}}{b_i^{m_i}} + \frac{1}{b_i^{m_i}}$$

for all $1 \le i \le s$. This, however, is equivalent to $n_0^{(i)} = l_{i,0}, \ldots, n_{m_i-1}^{(i)} = l_{i,m_i-1}$ which in turn is equivalent to $n \equiv l_{i,0} + l_{i,1} b_i + \cdots + l_{i,m_i-1} b_i^{m_i-1} \pmod{b_i^{m_i}}$ for all $1 \le i \le s$.

As $b_1, \ldots, b_s$ are pairwise relatively prime we obtain from the Chinese Remainder Theorem that among every $b_1^{m_1} \cdots b_s^{m_s}$ consecutive elements of the van der Corput-Halton sequence exactly one element is contained in $J_{\boldsymbol{l}}$ or, in other words, $A(J_{\boldsymbol{l}}, t b_1^{m_1} \cdots b_s^{m_s}, \mathcal{S}) = t$ for all $t \in \mathbb{N}$ and hence

$$A(J_{\boldsymbol{l}}, t b_1^{m_1} \cdots b_s^{m_s}, \mathcal{S}) - t b_1^{m_1} \cdots b_s^{m_s} \lambda_s(J_{\boldsymbol{l}}) = 0.$$

Therefore, for every $N \in \mathbb{N}$ we obtain

$$|A(J_{\boldsymbol{l}}, N, \mathcal{S}) - N \lambda_s(J_{\boldsymbol{l}})| \le 1.$$

Now we write the interval J as a disjoint union of intervals of the form $J_{\boldsymbol{l}}$,

$$J = \bigcup_{l_1=u_1}^{v_1-1} \cdots \bigcup_{l_s=u_s}^{v_s-1} J_{\boldsymbol{l}},$$

where $\boldsymbol{l} = (l_1, \ldots, l_s)$. Then we have

$$|A(J, N, \mathcal{S}) - N \lambda_s(J)| \le \sum_{l_1=u_1}^{v_1-1} \cdots \sum_{l_s=u_s}^{v_s-1} |A(J_{\boldsymbol{l}}, N, \mathcal{S}) - N \lambda_s(J_{\boldsymbol{l}})| \le \prod_{i=1}^{s} (v_i - u_i),$$

which proves the first assertion.

For $N \le b_1^{m_1} \cdots b_s^{m_s}$ we have $A(J_{\boldsymbol{l}}, N, \mathcal{S}) \le 1$ for all $\boldsymbol{l} = (l_1, \ldots, l_s) \in \mathbb{N}_0^s$ with $0 \le l_i < b_i^{m_i}$ for $1 \le i \le s$ and hence

$$A(J, N, \mathcal{S}) = \sum_{l_1=u_1}^{v_1-1} \cdots \sum_{l_s=u_s}^{v_s-1} A(J_{\boldsymbol{l}}, N, \mathcal{S}) \le \prod_{i=1}^{s} (v_i - u_i).$$

This was the second assertion of the lemma. □

Lemma 3.38 *Let $k \in \mathbb{N}$ and let $b_1, \dots, b_k \geq 2$ be integers. For $N \in \mathbb{N}$ let $d(b_1, \dots, b_k; N)$ be the number of tuples $(j_1, \dots, j_k) \in \mathbb{N}^k$ such that $b_1^{j_1} \cdots b_k^{j_k} \leq N$. Then we have*

$$d(b_1, \dots, b_k; N) \leq \frac{1}{k!} \prod_{i=1}^{k} \frac{\log N}{\log b_i}.$$

Proof Assume that $\boldsymbol{j} = (j_1, \dots, j_k) \in \mathbb{N}^k$ satisfies $b_1^{j_1} \cdots b_k^{j_k} \leq N$. Then the interval $E_{\boldsymbol{j}} := \prod_{i=1}^{k} [j_i - 1, j_i)$ of volume one is entirely contained in the simplex

$$S := \{(x_1, \dots, x_k) \in [0, \infty)^k \, : \, x_1 \log b_1 + \cdots + x_k \log b_k \leq \log N\}$$

of volume $\frac{1}{k!} \prod_{i=1}^{s} \frac{\log N}{\log b_i}$. Hence

$$d(b_1, \dots, b_k; N) = \lambda_k \left(\bigcup_{E_{\boldsymbol{j}} \subseteq S} E_{\boldsymbol{j}} \right) \leq \lambda_k(S) = \frac{1}{k!} \prod_{i=1}^{s} \frac{\log N}{\log b_i}. \qquad \square$$

Lemma 3.39 *Let $N \in \mathbb{N}$ and let $b_1, \dots, b_k \geq 2$ be integers. Furthermore, for $1 \leq i \leq k$ let $c_0^{(i)}, c_1^{(i)}, \dots \geq 0$ be given such that $c_0^{(i)} \leq 1$ and $c_j^{(i)} \leq f_i$ for all $j \geq 1$ and all $1 \leq i \leq k$. Then we have*

$$\sum_{\substack{(j_1, \dots, j_k) \in \mathbb{N}_0^k \\ b_1^{j_1} \cdots b_k^{j_k} \leq N}} \prod_{i=1}^{k} c_{j_i}^{(i)} \leq \frac{1}{k!} \prod_{i=1}^{k} \left(f_i \frac{\log N}{\log b_i} + k \right).$$

Proof Let $\mathfrak{u} \subseteq \{1, \dots, k\}$. Then the number of k-tuples $(j_1, \dots, j_k)$ with $j_i > 0$ if $i \in \mathfrak{u}$, $j_i = 0$ if $i \notin \mathfrak{u}$ and $\prod_{i \in \mathfrak{u}} b_i^{j_i} \leq N$ is by Lemma 3.38 bounded above by $\frac{1}{|\mathfrak{u}|!} \prod_{i \in \mathfrak{u}} \frac{\log N}{\log b_i}$. Furthermore, each of these k-tuples contributes at most $\prod_{i \in \mathfrak{u}} f_i$ to the sum on the left hand side in the statement of the lemma. From this, and invoking the inequality $\frac{1}{|\mathfrak{u}|!} \leq \frac{k^{k-|\mathfrak{u}|}}{k!}$, we obtain

$$\begin{aligned} \sum_{\substack{(j_1, \dots, j_k) \in \mathbb{N}_0^k \\ b_1^{j_1} \cdots b_k^{j_k} \leq N}} \prod_{i=1}^{k} c_{j_i}^{(i)} &\leq \sum_{\mathfrak{u} \subseteq \{1, \dots, k\}} \frac{1}{|\mathfrak{u}|!} \prod_{i \in \mathfrak{u}} f_i \frac{\log N}{\log b_i} \\ &\leq \frac{1}{k!} \sum_{\mathfrak{u} \subseteq \{1, \dots, k\}} k^{k-|\mathfrak{u}|} \prod_{i \in \mathfrak{u}} f_i \frac{\log N}{\log b_i} = \frac{1}{k!} \prod_{i=1}^{k} \left(f_i \frac{\log N}{\log b_i} + k \right), \end{aligned}$$

and this is the desired result. $\square$

Now we need to introduce some notation: let $J \subseteq \mathbb{R}^s$ be an interval. Then a *signed splitting* of J is a collection of not necessarily disjoint intervals $J_1, \ldots, J_r$ together with signs $\varepsilon_1, \ldots, \varepsilon_r \in \{-1, 1\}$ such that for all $x \in J$ we have

$$\sum_{\substack{i=1 \\ x \in J_i}}^{r} \varepsilon_i = 1.$$

As a consequence, for any additive function ν on the class of intervals in $\mathbb{R}^s$ we have

$$\nu(J) = \sum_{i=1}^{r} \varepsilon_i \nu(J_i).$$

Here, as usual, a function ν on the class of intervals in $\mathbb{R}^s$ is said to be additive if, whenever A and B are disjoint intervals, then $\nu(A \cup B) = \nu(A) + \nu(B)$.

Lemma 3.40 *Let $J = \prod_{i=1}^{s} [0, z_i)$ be an s-dimensional interval and let for each $1 \leq i \leq s$ be given a finite sequence $(z_{j,i})_{j=1,\ldots,n_i}$ of numbers in $[0,1]$. Define further $z_{0,i} := 0$ and $z_{n_i+1,i} := z_i$ for all $1 \leq i \leq s$. Then the collection of intervals*

$$\prod_{i=1}^{s} \left[\min\left(z_{j_i,i}, z_{j_i+1,i}\right), \max\left(z_{j_i,i}, z_{j_i+1,i}\right)\right)$$

together with the signs $\varepsilon_{j_1,\ldots,j_s} = \prod_{i=1}^{s} \operatorname{sgn}(z_{j_i+1,i} - z_{j_i,i})$ for $0 \leq j_i \leq n_i$ and $1 \leq i \leq s$ defines a signed splitting of the interval J.

Proof First we show the result for $s = 1$. For simplicity we omit the index i for the dimension. Let $J = [0, z)$ and let $z_0, \ldots, z_{n+1} \in [0, 1)$ with $z_0 = 0$ and $z_{n+1} = z$. Assume we are given a point $x \in [0, z)$. If $z_j \leq x$ for all $j = 0, \ldots, n+1$, then it follows that $x \notin J$. Now we define finite sequences $\underline{j}_k$, $k = 0, \ldots, K$ and $\overline{j}_k$, $k = 0, \ldots, K-1$ in the following way: let $\overline{j}_0 > 0$ be minimal, such that $z_{\overline{j}_0} > x$ and let $\underline{j}_0 > \overline{j}_0$ be minimal such that $z_{\underline{j}_0} \leq x$, let $\overline{j}_1 > \underline{j}_0$ be minimal, such that $z_{\overline{j}_1} > x$ and let $\underline{j}_1 > \overline{j}_1$ be minimal such that $z_{\underline{j}_1} \leq x$. We repeat this procedure and finally we choose $\overline{j}_K > \underline{j}_{K-1}$ to be minimal, such that $z_{\overline{j}_K} > x$ and $z_j > x$ for all $j \geq \overline{j}_K$. Since $z_{n+1} = z > x$, we always end in such a case.

With this definition we have $z_{\overline{j}_k - 1} \leq x < z_{\overline{j}_k}$ for $k = 0, \ldots, K$ and $z_{\underline{j}_k} \leq x < z_{\underline{j}_k - 1}$ for $k = 0, \ldots, K-1$

For $J_j := [\min(z_j, z_{j+1}), \max(z_j, z_{j+1}))$ and $\varepsilon_j := \operatorname{sgn}(z_{j+1} - z_j)$ we then

have $x \in J_{\overline{j}_k - 1}$ with $\varepsilon_{\overline{j}_k - 1} = +1$ and $x \in J_{\underline{j}_k - 1}$ with $\varepsilon_{\underline{j}_k - 1} = -1$ and $x \notin J_j$ for $j \neq \underline{j}_k - 1$ or $j \neq \overline{j}_k - 1$. Hence

$$\sum_{\substack{j=0 \\ x \in J_j}}^{n} \varepsilon_j = \sum_{\substack{j=0 \\ z_j \le x < z_{j+1}}}^{n} 1 - \sum_{\substack{j=0 \\ z_{j+1} \le x < z_j}}^{n} 1 = \sum_{k=0}^{K} 1 - \sum_{k=0}^{K-1} 1 = 1,$$

and thus we have a signed splitting of the interval $J = [0, z)$.

Now we turn to the multi-dimensional case: assume we are given a point $\boldsymbol{x} = (x_1, \ldots, x_s) \in J$. Then we have

$$\boldsymbol{x} \in J_{j_1,\ldots,j_s} = \prod_{i=1}^{s} J_{j_i,i} := \prod_{i=1}^{s} \left[\min\left(z_{j_i,i}, z_{j_i+1,i}\right), \max\left(z_{j_i,i}, z_{j_i+1,i}\right)\right),$$

if and only if $x_i \in J_{j_i,i}$ for all $1 \le i \le s$. Then we have

$$\underbrace{\sum_{j_1=0}^{n_1} \cdots \sum_{j_s=0}^{n_s}}_{\boldsymbol{x} \in J_{j_1,\ldots,j_s}} \varepsilon_{j_1,\ldots,j_s} = \prod_{i=1}^{s} \sum_{\substack{j_i=0 \\ x_i \in J_{j_i,i}}}^{n_i} \operatorname{sgn}(z_{j_i+1,i} - z_{j_i,i}) = 1,$$

where the last equality follows from the fact that for each $1 \le i \le s$ the collection of intervals $J_{j_i,i}$ together with the signs $\varepsilon_{j_i} = \operatorname{sgn}(z_{j_i+1,i} - z_{j_i,i})$ for $0 \le j_i \le n_i$, defines a signed splitting of the interval $[0, z_i)$ as shown above. □

For the proof of Theorem 3.36 we need a digit expansion of reals $z \in [0, 1)$ in an integer base $b \ge 2$ which uses signed digits. The next lemma shows that such an expansion exists.

Lemma 3.41 *Let $b \ge 2$ be an integer. Then every $z \in [0, 1)$ can be written in the form*

$$z = a_0 + \frac{a_1}{b} + \frac{a_2}{b^2} + \cdots$$

with integer digits $a_0, a_1, a_2, \ldots$ such that $-\left\lfloor \frac{b-1}{2} \right\rfloor \le a_j \le \left\lfloor \frac{b}{2} \right\rfloor$ for all $j \in \mathbb{N}_0$. This expansion is called the signed b-adic digit expansion of z.

Proof For $b = 2$ we may use the usual b-adic digit expansion. For $b \ge 3$ let $c = \lfloor \frac{b-1}{2} \rfloor$ and $x = cb^{-1} + cb^{-2} + cb^{-3} + \cdots \in [0, 1)$. For $z \in [0, 1)$ we have $z + x \in [0, 2)$ with b-adic expansion $z + x = u_0 + u_1 b^{-1} + u_2 b^{-2} + \cdots$, where $u_0 \in \{0, 1\}$ and $u_1, u_2, \ldots \in \{0, \ldots, b-1\}$. Hence

$$z = u_0 + \frac{u_1 - c}{b} + \frac{u_2 - c}{b^2} + \cdots$$

with $-\lfloor\frac{b-1}{2}\rfloor \le 0 \le u_0 \le 1 \le \lfloor\frac{b}{2}\rfloor$ and $-\lfloor\frac{b-1}{2}\rfloor \le u_j - c \le b-1-\lfloor\frac{b-1}{2}\rfloor = \lfloor\frac{b}{2}\rfloor$ for $j \in \mathbb{N}$. □

Proof of Theorem 3.36 Let $J = [\mathbf{0}, \boldsymbol{z}) \subseteq [0,1)^s$ with $\boldsymbol{z} = (z_1, \ldots, z_s)$. According to Lemma 3.41 for all $1 \le i \le s$ we consider the signed b_i-adic digit expansion of z_i of the form $z_i = a_{i,0} + a_{i,1}b_i^{-1} + a_{i,2}b_i^{-2} + a_{i,3}b_i^{-3} + \cdots$ with $-\lfloor(b_i-1)/2\rfloor \le a_{i,j} \le \lfloor b_i/2\rfloor$.

For all $1 \le i \le s$ let $n_i := \lfloor \log N / \log b_i \rfloor$ and for $1 \le l \le n_i$ define the truncations of the expansions $z_{l,i} = \sum_{j=0}^{l-1} a_{i,j}b_i^{-j}$ and let $z_{0,i} = 0$ and $z_{n_i+1,i} = z_i$.

According to Lemma 3.40 the collection of intervals

$$J_{\boldsymbol{j}} = \prod_{i=1}^{s} \left[\min\left(z_{j_i,i}, z_{j_i+1,i}\right), \max\left(z_{j_i,i}, z_{j_i+1,i}\right)\right)$$

together with the signs $\varepsilon_{\boldsymbol{j}} = \prod_{i=1}^{s} \operatorname{sgn}(z_{j_i+1,i} - z_{j_i,i})$ for $\boldsymbol{j} = (j_1, \ldots, j_s)$ and $0 \le j_i \le n_i$, $1 \le i \le s$ defines a signed splitting of the interval $J = [\mathbf{0}, \boldsymbol{z})$.

Since both, λ_s and $A(\cdot, N, \mathcal{S})$ are additive functions on the set of intervals, we obtain

$$A(J, N, \mathcal{S}) - N\lambda_s(J) = \sum_{j_1=0}^{n_1} \cdots \sum_{j_s=0}^{n_s} \varepsilon_{\boldsymbol{j}} (A(J_{\boldsymbol{j}}, N, \mathcal{S}) - N\lambda_s(J_{\boldsymbol{j}})) =: \Sigma_1 + \Sigma_2,$$

where Σ_1 denotes the sum over all $\boldsymbol{j} = (j_1, \ldots, j_s)$ such that $b_1^{j_1} \cdots b_s^{j_s} \le N$ and Σ_2 denotes the remaining part of the above sum.

First we deal with the sum Σ_1. For any $1 \le i \le s$ the length of the interval $[\min(z_{j_i,i}, z_{j_i+1,i}), \max(z_{j_i,i}, z_{j_i+1,i}))$ is $|a_{i,j_i} b_i^{-j_i}|$ and also the limit points of this interval are rationals with denominator $b_i^{j_i}$. Hence, the intervals $J_{\boldsymbol{j}}$ are of the form as considered in Lemma 3.37 from which we obtain now

$$|A(J_{\boldsymbol{j}}, N, \mathcal{S}) - N\lambda_s(J_{\boldsymbol{j}})| \le \prod_{i=1}^{s} |a_{i,j_i}|.$$

We have $|a_{i,j_i}| \le \lfloor b_i/2\rfloor =: f_i$. An application of Lemma 3.39 yields then

$$\Sigma_1 \le \frac{1}{s!} \prod_{i=1}^{s} \left(\frac{\lfloor b_i/2\rfloor \log N}{\log b_i} + s \right).$$

It remains to estimate Σ_2. To this end we split the set of s-tuples $\boldsymbol{j} = (j_1, \ldots, j_s)$ for which $b_1^{j_1} \cdots b_s^{j_s} > N$ into disjoint sets $B_0, \ldots, B_{s-1}$, where, for $1 \le k \le s-1$, we set

$$B_k = \{\boldsymbol{j} \in \mathbb{N}_0^s : b_1^{j_1} \cdots b_k^{j_k} \le N \text{ and } b_1^{j_1} \cdots b_k^{j_k} b_{k+1}^{j_{k+1}} > N\}$$

and $B_0 = \{\boldsymbol{j} \in \mathbb{N}_0^s : b_1^{j_1} > N\}$.

For a fixed $0 \le k \le s-1$ and a fixed k-tuple $(j_1, \dots, j_k)$ with $b_1^{j_1} \cdots b_k^{j_k} \le N$ define r to be the largest integer such that $b_1^{j_1} \cdots b_k^{j_k} b_{k+1}^{r-1} \le N$. Then the tuple $(j_1, \dots, j_k, j_{k+1}, \dots, j_s)$ is contained in B_k, if and only if $j_{k+1} \ge r$ (and $j_{k+2}, \dots, j_s$ can be chosen arbitrarily).

Therefore, for any $k \ge 0$ and fixed $j_1, \dots, j_k \in \mathbb{N}_0$ such that $b_1^{j_1} \cdots b_k^{j_k} \le N$, we have

$$\sum_{\substack{j_{k+1}, \dots, j_s \in \mathbb{N} \\ \boldsymbol{j} \in B_k}} \varepsilon_{\boldsymbol{j}} (A(J_{\boldsymbol{j}}, N, \mathcal{S}) - N\lambda_s(J_{\boldsymbol{j}})) = \pm (A(K, N, \mathcal{S}) - N\lambda_s(K)),$$

where

$$K = \prod_{i=1}^{k} \left[\min\left(z_{j_i,i}, z_{j_{i+1},i}\right), \max\left(z_{j_i,i}, z_{j_{i+1},i}\right)\right) \times \left[\min\left(z_{r,k+1}, z_{k+1}\right), \max\left(z_{r,k+1}, z_{k+1}\right)\right) \times \prod_{i=k+2}^{s} [0, z_i).$$

Let $\boldsymbol{j} \in B_k$. As

$$|z_{k+1} - z_{r,k+1}| \le \left\lfloor \frac{b_{k+1}}{2} \right\rfloor \frac{1}{b_{k+1}^r} \frac{b_{k+1}}{b_{k+1} - 1} \le \frac{1}{b_{k+1}^{r-1}}$$

it follows that the interval $\left[\min\left(z_{r,k+1}, z_{k+1}\right), \max\left(z_{r,k+1}, z_{k+1}\right)\right)$ is contained in some interval $\left[m_1/b_{k+1}^r, m_2/b_{k+1}^r\right)$ for $m_1, m_2 \in \mathbb{N}_0$ and with $m_2 - m_1 \le b_{k+1}$ and hence K is contained in the interval

$$K' = \prod_{i=1}^{k} \left[\min\left(z_{j_i,i}, z_{j_{i+1},i}\right), \max\left(z_{j_i,i}, z_{j_{i+1},i}\right)\right) \times \left[\frac{m_1}{b_{k+1}^r}, \frac{m_2}{b_{k+1}^r}\right) \times [0,1)^{s-k-1}.$$

Note that $\boldsymbol{j} \in B_k$ and hence $N < b_1^{j_1} \cdots b_k^{j_k} b_{k+1}^r$. Thus, an application of Lemma 3.37 yields

$$A(K, N, \mathcal{S}) \le A(K', N, \mathcal{S}) \le b_{k+1} \prod_{i=1}^{k} |a_{i,j_i}|.$$

But on the other hand we also have $N\lambda_s(K) \le b_{k+1} \prod_{i=1}^{k} |a_{i,j_i}|$ and hence

$$|A(K, N, \mathcal{S}) - N\lambda_s(K)| \le b_{k+1} \prod_{i=1}^{k} |a_{i,j_i}| \le b_{k+1} \prod_{i=1}^{k} c_{i,j_i},$$

where $c_{i,j_i} = 1$ if $j_i = 0$ and $c_{i,j_i} = \lfloor b_i/2 \rfloor$ otherwise.

Summing up we obtain

$$|\Sigma_2| \le \sum_{k=0}^{s-1} \sum_{\substack{j_1,\ldots,j_k \in \mathbb{N}_0 \\ b_1^{j_1}\cdots b_k^{j_k} \le N}} \left| \sum_{\boldsymbol{j} \in B_k} \varepsilon_{\boldsymbol{j}} (A(J_{\boldsymbol{j}}, N, \mathcal{S}) - N\lambda_s(J_{\boldsymbol{j}})) \right|$$

$$\le \sum_{k=0}^{s-1} \sum_{\substack{j_1,\ldots,j_k \in \mathbb{N}_0 \\ b_1^{j_1}\cdots b_k^{j_k} \le N}} b_{k+1} \prod_{i=1}^{k} c_{i,j_i} \le \sum_{k=0}^{s-1} \frac{b_{k+1}}{k!} \prod_{i=1}^{k} \left(\frac{\lfloor b_i/2 \rfloor \log N}{\log b_i} + k \right),$$

where we used Lemma 3.39 again. Hence the result follows. □

Corollary 3.42 *Let $b_1, \ldots, b_s \ge 2$ be pairwise relatively prime integers and let $\mathcal{S}$ be the van der Corput-Halton sequence with bases $b_1, \ldots, b_s$. Then for any $N \ge 2$ we have*

$$D_N^*(\mathcal{S}) \le c(b_1, \ldots, b_s) \frac{(\log N)^s}{N} + O\left(\frac{(\log N)^{s-1}}{N} \right),$$

with

$$c(b_1, \ldots, b_s) = \frac{1}{s!} \prod_{i=1}^{s} \frac{\lfloor b_i/2 \rfloor}{\log b_i}.$$

Furthermore, if $b_1, \ldots, b_s$ are the first s primes, then $c(b_1, \ldots, b_s) \le \frac{7}{2^s s}$.

Proof The first part of the corollary follows immediately from Theorem 3.36. Hence let us assume that $b_1, \ldots, b_s$ are the first s prime numbers in increasing order. Then $b_2, \ldots, b_s$ are odd and hence $\lfloor b_i/2 \rfloor = (b_i - 1)/2$ for $2 \le i \le s$. Let $\pi(x)$ denote the prime counting function, i.e., $\pi(x)$ counts all prime numbers less than or equal to x. For any $x \ge 11$ we have $\pi(x) > x/\log x$; see [229, Chapter VII]. Therefore we find that for $i \ge 6$ we have $i - 1 = \pi(b_i - 1) > \frac{b_i - 1}{\log(b_i - 1)} > \frac{b_i - 1}{\log b_i}$. Consequently, for $i \ge 6$, we have $\frac{b_i - 1}{i \log b_i} < \frac{i-1}{i}$ and hence, for $s \ge 6$ we have

$$c(b_1, \ldots, b_s) \le \frac{2^5 A}{2^s} \prod_{i=6}^{s} \frac{i-1}{i}$$

where $A = \frac{2 \cdot 3 \cdot 5}{5! \log 2 \cdot \log 3 \cdot \log 5 \cdot \log 7 \cdot \log 11}$. Since $2^5 A \prod_{i=6}^{s} \frac{i-1}{i} = 2^5 A \frac{5}{s} < \frac{7}{s}$ it follows that

$$c(b_1, \ldots, b_s) \le \frac{7}{2^s s}$$

for all $s \ge 6$. The bound $c(b_1, \ldots, b_s) \le \frac{7}{2^s s}$ for $1 \le s \le 5$ can be shown numerically. □

Remark 3.43 For $s = 1$ we have the one-dimensional van der Corput sequence $\mathcal{S}$ as introduced in Definition 3.10. In this case Faure [65] proved that

$$\limsup_{N\to\infty} \frac{ND_N^*(\mathcal{S})}{\log N} = \begin{cases} \frac{b-1}{4\log b} & \text{if } b \text{ is odd,} \\ \frac{b^2}{4(b+1)\log b} & \text{if } b \text{ is even.} \end{cases}$$

For more exact results in the case $b = 2$ we refer to [13]. A Central Limit Theorem for the star discrepancy of the van der Corput sequence in base 2 can be found in [60, Theorem 2]. Concerning results on the star discrepancy of generalisations of the one-dimensional van der Corput sequence we refer, for example, to [65, 68, 69, 70, 71, 125].

Based on the (infinite) $(s-1)$-dimensional van der Corput Halton sequence one can introduce a finite s-dimensional point set which is known as Hammersley point set.

Definition 3.44 For dimensions $s \geq 2$ the *Hammersley point set* with integer bases $b_1, \ldots, b_{s-1} \geq 2$ consisting of $N \in \mathbb{N}$ points in the s-dimensional unit cube is the point set $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$ where the nth element is given by $\boldsymbol{x}_n = \big(n/N, \varphi_{b_1}(n), \ldots, \varphi_{b_{s-1}}(n)\big)$ for $0 \leq n \leq N-1$.

We deduce a discrepancy bound for the Hammersley point set with the help of Theorem 3.36 in combination with the following general result that goes back to Roth [226] (see also [175, Lemma 3.7]).

Lemma 3.45 *For $s \geq 2$ let $\mathcal{S} = (\boldsymbol{y}_n)_{n\geq 0}$, where $\boldsymbol{y}_n = (y_{n,1}, \ldots, y_{n,s-1})$ for $n \geq 0$, be an arbitrary sequence in the $(s-1)$-dimensional unit cube with star discrepancy $D_N^*(\mathcal{S})$. For $N \in \mathbb{N}$ consider the point set $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$ in the s-dimensional unit cube given by $\boldsymbol{x}_n = (n/N, y_{n,1}, \ldots, y_{n,s-1})$ for $0 \leq n \leq N-1$ with star discrepancy $D_N^*(\mathcal{P})$. Then we have*

$$D_N^*(\mathcal{P}) \leq \frac{1}{N}\left(\max_{1\leq m\leq N} mD_m^*(\mathcal{S}) + 1\right).$$

Proof Consider a sub-interval of the s-dimensional unit cube of the form $E = \prod_{i=1}^s [0, u_i)$. Then a point $\boldsymbol{x}_n$, $0 \leq n \leq N-1$, belongs to E, if and only if $0 \leq n < Nu_1$ and $\boldsymbol{y}_n \in \prod_{i=2}^s [0, u_i)$. Denoting $E' = \prod_{i=2}^s [0, u_i)$ we have $A(E, N, \mathcal{P}) = A(E', m, \mathcal{S})$ with $m := \lceil Nu_1 \rceil$ and therefore

$$|A(E,N,\mathcal{P}) - N\lambda_s(E)| \leq |A(E',m,\mathcal{S}) - m\lambda_{s-1}(E')| + |m\lambda_{s-1}(E') - N\lambda_s(E)|.$$

We have $|m\lambda_{s-1}(E') - N\lambda_s(E)| \leq |(\lceil Nu_1 \rceil - Nu_1)\prod_{i=2}^s u_i| \leq 1$ and hence

$$|A(E,N,\mathcal{P}) - N\lambda_s(E)| \leq mD_m^*(\mathcal{S}) + 1$$

and the result follows. □

Now we can give an estimate for the star discrepancy of the Hammersley point set. The proof of the subsequent result follows directly from Theorem 3.36 and Lemma 3.45.

Theorem 3.46 *Let $b_1, \ldots, b_{s-1} \geq 2$ be pairwise relatively prime integers and let $N \in \mathbb{N}$. Then the star discrepancy of the Hammersley point set $\mathcal{P}$ with bases $b_1, \ldots, b_{s-1}$ consisting of N points in the s-dimensional unit cube is bounded by*

$$ND_N^*(\mathcal{P}) \leq \frac{1}{(s-1)!} \prod_{i=1}^{s-1} \left(\frac{\lfloor b_i/2 \rfloor \log N}{\log b_i} + s - 1 \right) + \sum_{k=0}^{s-2} \frac{b_{k+1}}{k!} \prod_{i=1}^{k} \left(\frac{\lfloor b_i/2 \rfloor \log N}{\log b_i} + k \right) + 1.$$

It follows from Theorem 3.46 that for the star discrepancy of the s-dimensional Hammersley point set $\mathcal{P}$ in pairwise relatively prime bases $b_1, \ldots, b_{s-1}$ consisting of N points we have

$$D_N^*(\mathcal{P}) \leq c(b_1, \ldots, b_{s-1}) \frac{(\log N)^{s-1}}{N} + O\left(\frac{(\log N)^{s-2}}{N} \right),$$

where $c(b_1, \ldots, b_{s-1}) = \frac{1}{(s-1)!} \prod_{i=1}^{s-1} \frac{\lfloor b_i/2 \rfloor}{\log b_i}$. In the case that $b_1, \ldots, b_{s-1}$ are the first $s-1$ prime numbers we have $c(b_1, \ldots, b_{s-1}) \leq \frac{7}{2^{s-1}(s-1)}$.

An exact formula for the star discrepancy of the two-dimensional Hammersley point set can be found in [32], see also [88, 142] for the base 2 case and [67, Theorem 1] for a bound and [72] for exact results on the star discrepancy of generalised versions of the two-dimensional Hammersley point set.

Informally, one calls a point set consisting of N points in the s-dimensional unit cube a *low-discrepancy point set*, if its star discrepancy is of order $(\log N)^{s-1}/N$. In this sense the Hammersley point set in pairwise relatively prime bases is a low-discrepancy point set. Recall that it is widely believed that this order is the best possible for the star discrepancy of a finite point set.

Lattice point sets

Now we turn to a further construction of finite point sets with low star discrepancy which is often called the method of good lattice points. Those point sets originated independently by Hlawka [111] and Korobov [121].

Definition 3.47 Let $\boldsymbol{g} \in \mathbb{N}^s$ and let $N \in \mathbb{N}$. A point set $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$ in the s-dimensional unit cube with $\boldsymbol{x}_n = \{n\boldsymbol{g}/N\}$ for all $0 \le n \le N-1$ is called *lattice point set* and $\boldsymbol{g}$ is called the *generating vector* of the lattice point set. A QMC rule that uses a lattice point set as underlying quadrature points is often called a *lattice rule*.

Example 3.48 For example if we choose $N = 34$ and $\boldsymbol{g} = (1, 21)$, then we obtain the point set shown in the left picture of Figure 3.6. For $N = 144$ and $\boldsymbol{g} = (1, 89)$ we obtain the point set shown on the right of Figure 3.6.

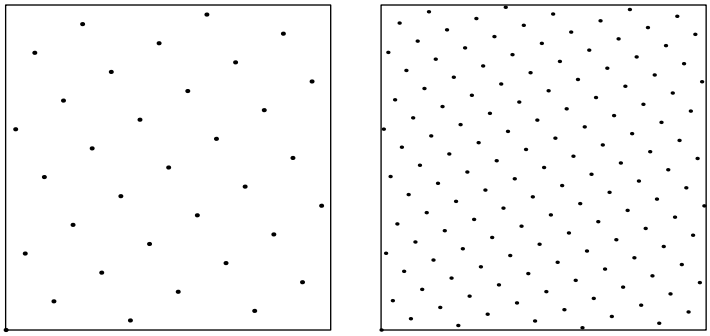

Figure 3.6 Two-dimensional lattice point sets with $N = 34$ and $\boldsymbol{g} = (1, 21)$ (left picture) and with $N = 144$ and $\boldsymbol{g} = (1, 89)$ (right picture).

For a lattice point set $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$ consisting of N points and with generating vector $\boldsymbol{g} \in \mathbb{N}^s$ we have that each point $\boldsymbol{x}_n$ is of the form $\boldsymbol{x}_n = \{\boldsymbol{y}_n/N\}$ with $\boldsymbol{y}_n = n\boldsymbol{g} \in \mathbb{Z}^s$. Hence we can apply Theorem 3.27 from which we obtain

$$D_N(\mathcal{P}) \le 1 - \left(1 - \frac{1}{N}\right)^s + \sum_{\boldsymbol{h} \in C_s^*(N)} \frac{1}{r(\boldsymbol{h}, N)} \left| \frac{1}{N} \sum_{n=0}^{N-1} \mathrm{e}^{2\pi \mathrm{i} n \boldsymbol{h} \cdot \boldsymbol{g}/N} \right|.$$

Using the formula for a geometric sum we obtain

$$\sum_{n=0}^{N-1} \mathrm{e}^{2\pi \mathrm{i} n \boldsymbol{h} \cdot \boldsymbol{g}/N} = \begin{cases} N & \text{if } \boldsymbol{h} \cdot \boldsymbol{g} \equiv 0 \pmod N, \\ 0 & \text{if } \boldsymbol{h} \cdot \boldsymbol{g} \not\equiv 0 \pmod N. \end{cases} \tag{3.15}$$

Furthermore, for $\boldsymbol{h} \in C_s^*(N)$ we have $r(\boldsymbol{h}, N) \ge 2r(\boldsymbol{h})$ where $r(\boldsymbol{h}) = \prod_{i=1}^s r(h_i)$ for $\boldsymbol{h} = (h_1, \ldots, h_s)$ and $r(h) = \max(1, |h|)$. This follows from the fact that $\sin(\pi t) \ge 2t$ for $0 \le t \le \frac{1}{2}$. Altogether we obtain the following bound on the extreme discrepancy of a lattice point set.

Proposition 3.49 *For the extreme discrepancy of a lattice point set $\mathcal{P}$ consisting of N points and with generating vector $\boldsymbol{g} \in \mathbb{N}^s$ we have*

$$D_N(\mathcal{P}) \le \frac{s}{N} + \frac{1}{2} \sum_{\substack{\boldsymbol{h} \in C_s^*(N) \\ \boldsymbol{h}\cdot\boldsymbol{g} \equiv 0 \pmod N}} \frac{1}{r(\boldsymbol{h})}.$$

Starting from this bound one can show (see, for example, [175, Section 5]) by using an average argument that for every dimension s and every $N \in \mathbb{N}$ there exist generating vectors $\boldsymbol{g} = (g_1, \ldots, g_s)$ with $0 \le g_i < N$ and $\gcd(g_i, N) = 1$ for all $1 \le i \le s$ such that the corresponding lattice point set has extreme discrepancy of order $(\log N)^s/N$. Such a vector is often called a *good lattice point*. However, this result is by no means constructive, i.e, it is not known how a general construction principle for a good lattice point can be deduced from it. For a long time one had to rely on time consuming computer searches for good lattice points. A considerably faster search algorithm was introduced by Sloan & Reztsov [246] which allows one to find good lattice points in reasonably high dimension with a reasonably large number of points. This method is nowadays known as *component-by-component construction* or short *cbc-construction*. Here the basic idea is to start with a good one-dimensional lattice point and then one appends step by step a further dimension to the already constructed good lattice point such that also the new lattice point is a good one. Joe [113] was the first who used this approach for searching for lattice point sets with low discrepancy.

First we mention that by using (3.15), the sum which appeared in Proposition 3.49 can be written as

$$\begin{aligned} R_N(\boldsymbol{g}) &:= \sum_{\substack{\boldsymbol{h} \in C_s^*(N) \\ \boldsymbol{h}\cdot\boldsymbol{g} \equiv 0 \pmod N}} \frac{1}{r(\boldsymbol{h})} \\ &= -1 + \frac{1}{N} \sum_{n=0}^{N-1} \prod_{i=1}^{s} \left(1 + \sum_{\substack{-N/2 < h \le N/2 \\ h \ne 0}} \frac{\mathrm{e}^{2\pi \mathrm{i} h n g_i/N}}{|h|} \right). \end{aligned} \tag{3.16}$$

Therefore, for given dimension s, the calculation of $R_N(\boldsymbol{g})$ would require $O(N^2 s)$ operations which can be reduced to $O(Ns)$ operations by using an asymptotic expansion due to Joe & Sloan [116].

Now we use the following component-by-component algorithm for the construction of a good lattice point.

Algorithm 3.50 *Let $N \in \mathbb{N}$ and let $G_N = \{1, \ldots, N-1\}$.*

1. Choose $g_1 = 1$.

2. For $d > 1$, assume we have already constructed $g_1, \ldots, g_{d-1}$. Then find $g_d \in G_N$ which minimises $R_N((g_1, \ldots, g_{d-1}, g_d))$ as a function of g_d.

If N is a prime number, then one can show that Algorithm 3.50 provides a good lattice point. (For results concerning composite N we refer to [240].)

Theorem 3.51 *Let N be a prime number and suppose that $\boldsymbol{g} = (g_1, \ldots, g_s)$ is constructed according to Algorithm 3.50. Then for all $1 \le d \le s$ we have*

$$R_N((g_1, \ldots, g_d)) \le \frac{1}{N-1}(1 + S_N)^d,$$

where $S_N = \sum_{h \in C_1^*(N)} |h|^{-1}$.

Proof Since N is a prime number it follows that $R_N(g_1) = 0$ for all $g_1 \in G_N$. Let $d \ge 1$ and assume that we have

$$R_N(\boldsymbol{g}) \le \frac{1}{N-1}(1 + S_N)^d,$$

where $\boldsymbol{g} = (g_1, \ldots, g_d)$. Now we consider $(\boldsymbol{g}, g_{d+1}) := (g_1, \ldots, g_d, g_{d+1})$.

As g_{d+1} minimises $R_N((\boldsymbol{g}, \cdot))$ over G_N we obtain

$$\begin{aligned} &R_N((\boldsymbol{g}, g_{d+1})) \\ &\le \frac{1}{N-1} \sum_{g_{d+1}=1}^{N-1} \sum_{\substack{(\boldsymbol{h}, h_{d+1}) \in C_{d+1}^*(N) \\ \boldsymbol{h}\cdot\boldsymbol{g} + h_{d+1} g_{d+1} \equiv 0 \pmod N}} \frac{1}{r(\boldsymbol{h})} \frac{1}{r(h_{d+1})} \\ &= \sum_{(\boldsymbol{h}, h_{d+1}) \in C_{d+1}^*(N)} \frac{1}{r(\boldsymbol{h})} \frac{1}{r(h_{d+1})} \frac{1}{N-1} \sum_{\substack{g_{d+1} \in G_N \\ \boldsymbol{h}\cdot\boldsymbol{g} + h_{d+1} g_{d+1} \equiv 0 \pmod N}} 1, \end{aligned}$$

where we just changed the order of summation. Separating out the term where $h_{d+1} = 0$ we obtain

$$\begin{aligned} &R_N((\boldsymbol{g}, g_{d+1})) \\ &\le R_N(\boldsymbol{g}) + \sum_{\boldsymbol{h} \in C_d(N)} \frac{1}{r(\boldsymbol{h})} \sum_{h_{d+1} \in C_1^*(N)} \frac{1}{r(h_{d+1})} \frac{1}{N-1} \sum_{\substack{g_{d+1} \in G_N \\ h_{d+1} g_{d+1} \equiv -\boldsymbol{h}\cdot\boldsymbol{g} \pmod N}} 1. \end{aligned}$$

Since N is a prime, the congruence $h_{d+1} g_{d+1} \equiv -\boldsymbol{h} \cdot \boldsymbol{g} \pmod N$ has exactly one solution $g_{d+1} \in G_N$ if $\boldsymbol{h} \cdot \boldsymbol{g} \not\equiv 0 \pmod N$ and no solution in G_N if $\boldsymbol{h} \cdot \boldsymbol{g} \equiv 0 \pmod N$. From this insight it follows that

$$R_N((\boldsymbol{g}, g_{d+1})) \le R_N(\boldsymbol{g}) + \frac{1}{N-1} \sum_{\boldsymbol{h} \in C_d(N)} \frac{1}{r(\boldsymbol{h})} \sum_{h_{d+1} \in C_1^*(N)} \frac{1}{r(h_{d+1})}$$

$$
\begin{aligned}
&= R_N(\boldsymbol{g}) + \frac{S_N}{N-1} \sum_{\boldsymbol{h} \in C_d(N)} \frac{1}{r(\boldsymbol{h})} \\
&= R_N(\boldsymbol{g}) + \frac{S_N}{N-1}(1+S_N)^d \\
&\leq \frac{1}{N-1}(1+S_N)^d + \frac{S_N}{N-1}(1+S_N)^d \\
&= \frac{1}{N-1}(1+S_N)^{d+1},
\end{aligned}
$$

where we used the induction hypotheses to bound $R_N(\boldsymbol{g})$. This completes the proof of Theorem 3.51. □

It can be shown that $S_N \leq 2\log N + 2\gamma - \log 4 + 4N^{-2}$, where $\gamma = 0.577\ldots$ is the Euler constant (for a proof of this fact see [166, Lemmas 1 & 2]). Therefore, from Proposition 3.49 and Theorem 3.51 we obtain the following bound on the extreme discrepancy of the lattice point set whose generating vector is constructed with Algorithm 3.50.

Corollary 3.52 *Let N be a prime number and suppose that $\boldsymbol{g} = (g_1, \ldots, g_s)$ is constructed according to Algorithm 3.50. For $1 \leq d \leq s$ let $\mathcal{P}_d$ to denote the lattice point set generated by the lattice point $(g_1, \ldots, g_d)$. Then we have*

$$
D_N(\mathcal{P}_d) \leq \frac{d}{N} + \frac{1}{N}\left(2\log N + 2\gamma + 1 - \log 4 + \frac{4}{N^2}\right)^d.
$$

Hence, with Algorithm 3.50, one can construct a lattice point set in the s-dimensional unit cube whose extreme discrepancy is of order $(\log N)^s/N$. This is not quite as good as possible. For example, for the Hammersley point set we had an order of $(\log N)^{s-1}/N$. Nevertheless, the bound on $R_N(\boldsymbol{g})$ is best possible in the order of magnitude in N. This follows from a general lower bound due to Larcher [130], which states that for every $s \geq 2$ there exists a $c_s > 0$ such that for all $N \in \mathbb{N}$ and all lattice points $\boldsymbol{g}$ we have $R_N(\boldsymbol{g}) \geq c_s(\log N)^s/N$. For dimensions $s > 3$ it is still an open problem whether there are lattice point sets with discrepancy of order $(\log N)^{s-1}/N$. For dimension $s = 2$, such an order can be obtained with so-called Fibonacci lattice rules; see [175, Section 5].

Lattice point sets can have small extreme- and star discrepancy. However, one should mention that the full power of lattice point sets lies in QMC integration of smooth, one-periodic functions. For a detailed treatment of this topic we refer to [175, Section 5] or to [243].

3.5 Tractability of discrepancy

In many applications the dimension s can be rather large. But in this case, the asymptotically very good bounds on the discrepancy from the previous section are not useful already for modest cardinality N of a point set. For example, assume that for every $s, N \in \mathbb{N}$ we have a point set $\mathcal{P}_{s,N}$ in the s-dimensional unit cube of cardinality N with star discrepancy of at most

$$D_N^*(\mathcal{P}_{s,N}) \leq c_s \frac{(\log N)^s}{N},$$

for some $c_s > 0$. Hence for any $\varepsilon > 0$ the star discrepancy behaves asymptotically like $N^{-1+\varepsilon}$ which is of course excellent. However, the function $N \to (\log N)^s/N$ decreases to zero not until $N \geq \mathrm{e}^s$. For $N \leq \mathrm{e}^s$ this function is increasing which means that for cardinality N in this range our discrepancy bounds are useless. But already for moderately large dimension s, the value of e^s is huge, and even as huge, such that point sets with cardinality $N \geq \mathrm{e}^s$ cannot be used for practical applications. For example the case $s = 10$, which is not considered to be large in practical applications, is shown in Figure 3.7.

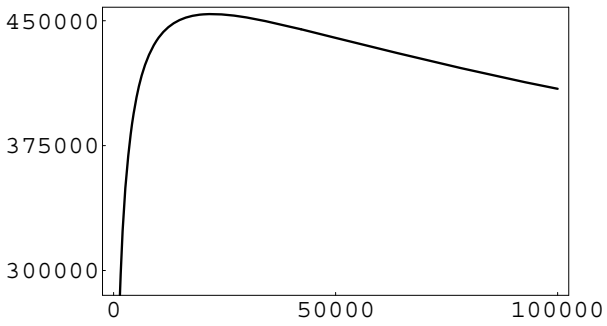

Figure 3.7 The function $N \mapsto (\log N)^s/N$ for $s = 10$.

Hence we are also interested in the discrepancy of point sets with not too large cardinality. To analyse this problem systematically we introduce the following quantity.

Definition 3.53 For integers $s, N \in \mathbb{N}$ let

$$\mathrm{disc}^*(N, s) = \inf_{\mathcal{P}} D_N^*(\mathcal{P}),$$

where the infimum is extended over all point sets $\mathcal{P}$ consisting of N points

in the s-dimensional unit cube. Then $\mathrm{disc}^*(N,s)$ is called the *Nth minimal star discrepancy*. Furthermore, for $\varepsilon > 0$ we define

$$N^*(s,\varepsilon) = \min\left\{N \in \mathbb{N} : \mathrm{disc}^*(N,s) \le \varepsilon\right\},$$

the so-called *inverse of star discrepancy*.

For example, consider point sets consisting of $N = 2^s$ points in the s-dimensional unit cube (for $s \ge 30$ this is already a huge cardinality). Can we say then that in each dimension s there exists a point set of such a cardinality such that its star discrepancy tends to zero as s grows to infinity? In terms of Definition 3.53 this would mean whether we can say that $\mathrm{disc}^*(2^s,s)$ goes to zero as $s \to \infty$ or not? But from the upper bounds on the star discrepancy of special point sets that we know so far, it is not known how to deduce an answer to such a question.

The best bounds on the star discrepancy that we know are all of asymptotic order $(\log N)^s/N$. If we insert here for the cardinality $N = 2^s$, then we obtain upper bounds of order

$$\frac{(\log 2^s)^s}{2^s} \approx \left(\frac{s}{2}\right)^s$$

which goes rapidly to infinity with s (and also the small constant $c_s = 7/(s2^s)$ from the star discrepancy bound for the van der Corput-Halton sequence cannot invert this behaviour).

As another example (see [196]) consider for an integer $m \ge 2$ the regular lattice Γ_m with $N = m^s$ points in the s-dimensional unit cube as defined in (3.14). From Proposition 3.32 we know that the star discrepancy of this point set is exactly

$$D_N^*(\Gamma_m) = 1 - \left(1 - \frac{1}{m}\right)^s.$$

Hence to obtain a star discrepancy of at most $\varepsilon > 0$ one needs a regular lattice with at least

$$\left(\frac{s}{|\log(1-\varepsilon)|}\right)^s$$

points. This number grows superexponentially in the dimension s. For example, N has to be at least $(1.45s)^s$ to obtain a star discrepancy smaller than one half.

Nevertheless, in spite of our negative results we found so far, the answer to the initially stated question whether there exist point sets consisting of $N = 2^s$ points in the s-dimensional unit cube whose star discrepancy tends to zero as s grows to infinity is *Yes*, and even much more is possible. This

was shown first by Heinrich, Novak, Wasilkowski & Woźniakowski [93]. They showed that there exists a constant $c > 0$ such that

$$\operatorname{disc}^*(N, s) \le c\sqrt{\frac{s}{N}} \tag{3.17}$$

for all $N, s \in \mathbb{N}$ from which it follows that

$$N^*(s, \varepsilon) \le Cs\varepsilon^{-2} \tag{3.18}$$

for some constant $C > 0$. Hence, the inverse of star discrepancy depends only polynomially on s and ε^{-1}. In Complexity Theory such a behaviour is called *polynomial tractability*.

Furthermore it is known that the dependence on the dimension s of the upper bound on the Nth minimal star discrepancy in (3.18) cannot be improved. It was shown by Hinrichs [107, Theorem 1] that there exist constants $c, \varepsilon_0 > 0$ such that $N^*(s, \varepsilon) \ge cs/\varepsilon$ for $0 < \varepsilon < \varepsilon_0$ and $\operatorname{disc}^*(N, s) \ge \min(\varepsilon_0, cs/n)$.

In comparison to (3.17) the law of the iterated logarithm for the star discrepancy (see [61, Theorem 1.193]) states that

$$\limsup_{N\to\infty} \frac{\sqrt{2N} D_N^*(\mathcal{S})}{\sqrt{\log\log N}} = 1$$

for almost all random sequences $\mathcal{S}$ in $[0, 1)^s$. However, this result gives absolutely no information about the dependence of the star discrepancy on the dimension s.

Here we show a slightly weaker bound than those given in (3.17). This result, which was also shown first in [93], has the advantage that its proof is more or less elementary and that it contains no unknown constants. Improvements can be found in [57, Theorem 3.2] (see also Exercise 3.28) and in [84, Theorem 2.1]. A similar result for the extreme discrepancy can be found in [84, Theorem 2.2].

Theorem 3.54 *For all $N, s \in \mathbb{N}$ we have*

$$\operatorname{disc}^*(N, s) \le \frac{2\sqrt{2}}{\sqrt{N}} \left(s \log\left(\left\lceil \frac{s\sqrt{N}}{2(\log 2)^{1/2}} \right\rceil + 1 \right) + \log 2 \right)^{1/2}. \tag{3.19}$$

For all $s \in \mathbb{N}$ and all $\varepsilon > 0$ we have

$$N^*(s, \varepsilon) \le \lceil 8\varepsilon^{-2}(s \log(\lceil 2s/\varepsilon \rceil + 1) + \log 2) \rceil. \tag{3.20}$$

Proof The proof is based on Hoeffding's inequality from Probability Theory which states the following: assume that $X_1, \ldots, X_n$ are independent random

variables with expectation 0 and $X_i \in [a_i, b_i]$ almost sure for $1 \le i \le n$, where $a_i < 0 < b_i$, and let $S_n := X_1 + \cdots + X_n$. Then for every $t \ge 0$ we have

$$\mathrm{Prob}\,[|S_n| \ge t] \le 2\mathrm{e}^{-2t^2/\sum_{i=1}^n (b_i - a_i)^2}.$$

Now let $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_N$ be independent, identically and uniformly on $[0,1)^s$ distributed random variables. For $\boldsymbol{x} = (x_1, \ldots, x_s)$ in $[0,1]^s$ and $1 \le i \le N$ let

$$\zeta_{\boldsymbol{x}}^{(i)} := \chi_{[\mathbf{0},\boldsymbol{x})}(\boldsymbol{\tau}_i) - x_1 \cdots x_s.$$

Then the expected value of $\zeta_{\boldsymbol{x}}^{(i)}$ is $\mathbb{E}[\zeta_{\boldsymbol{x}}^{(i)}] = 0$ and further we obviously have $|\zeta_{\boldsymbol{x}}^{(i)}| \le 1$ for all $1 \le i \le N$. Let $\delta > 0$. Using Höffding's inequality it follows that for all $\boldsymbol{x} \in [0,1]^s$ we have

$$\mathrm{Prob}\left[\left|\frac{1}{N}\sum_{i=1}^N \zeta_{\boldsymbol{x}}^{(i)}\right| \ge \delta\right] \le 2\mathrm{e}^{-\delta^2 N/2}.$$

Let Γ_m be the equidistant grid on $[0,1]^s$ with mesh-size $1/m$, where $m = \lceil s/\delta \rceil$. Using Proposition 3.17 we obtain now

$$\begin{aligned}
&\mathrm{Prob}\,[D_N^*(\{\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_N\}) \le 2\delta] \\
&\ge \mathrm{Prob}\left[\max_{\boldsymbol{x}\in\Gamma_m} \left|\frac{A([\mathbf{0},\boldsymbol{x}), N, \{\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_N\})}{N} - x_1 \cdots x_s\right| \le \delta\right] \\
&\ge 1 - 2(m+1)^s \mathrm{e}^{-\delta^2 N/2}.
\end{aligned}$$

The last expression is strictly larger than $c \ge 0$, if

$$\log \frac{2}{1-c} + s \log\left(\left\lceil \frac{s}{\delta} \right\rceil + 1\right) - \delta^2 \frac{N}{2} < 0. \tag{3.21}$$

This inequality holds for all $\delta > \delta_0 = \delta_0(N, s)$ where

$$\delta_0^2 = \frac{2}{N}\left(s \log\left(\left\lceil \frac{s}{\delta_0} \right\rceil + 1\right) + \log \frac{2}{1-c}\right). \tag{3.22}$$

Hence $\frac{1}{\delta_0} \le \left(\frac{N}{4 \log 2}\right)^{1/2}$ and substituting this result back into (3.22), it follows that

$$\delta_0^2 \le \frac{2}{N}\left(s \log\left(\left\lceil \frac{s\sqrt{N}}{2(\log 2)^{1/2}} \right\rceil + 1\right) + \log \frac{2}{1-c}\right).$$

Choosing $c = 0$, it follows that for all $\delta > \delta_0$ there exist $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_N \in [0,1)^s$ such that $D_N^*(\{\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_N\}) \le 2\delta_0$. Therefore we obtain (3.19).

We also have that there exist $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_N \in [0,1)^s$ with $D_N^*(\{\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_N\}) \le$

ε whenever inequality (3.21) with $c = 0$ is fulfilled with $\delta = \varepsilon/2$. This is the case for

$$N > 8\varepsilon^{-2}(s \log(\lfloor 2s/\varepsilon \rfloor + 1) + \log 2)$$

and hence (3.20) follows. □

Remark 3.55 From the proof of Theorem 3.54 we even obtain a little bit more. Namely, for $c \in [0, 1)$ we have that the probability to choose randomly a point set $\mathcal{P}$ consisting of N points in the s-dimensional unit cube with star discrepancy of at most

$$D_N^*(\mathcal{P}) \leq \frac{2\sqrt{2}}{\sqrt{N}} \left(s \log \left(\left\lceil \frac{s\sqrt{N}}{2(\log 2)^{1/2}} \right\rceil + 1 \right) + \log \frac{2}{1-c} \right)^{1/2}$$

is strictly larger than c.

The main disadvantage of Theorem 3.54 is that it is purely probabilistic and therefore by no means constructive. A first constructive approach is given in [57] which is further improved in [55]. Here a deterministic algorithm is presented that constructs point sets $\mathcal{P}_{N,s}$ consisting of N points in the s-dimensional unit cube satisfying

$$D_N^*(\mathcal{P}_{N,s}) = O\left(\frac{s^{1/2}}{N^{1/2}} (\log(N+1))^{1/2} \right)$$

in run-time $O(s \log(sN)(\sigma N)^s)$, where $\sigma = \sigma(s) = O((\log s)^2/(s \log\log s)) \to 0$ as $s \to \infty$ and where the implied constant in the O-notation is independent of s and N. This is by far too expensive for high dimensional applications. An implementation and numerical tests of the algorithm can be found in [59].

A further improvement is presented in [56]. Here a component-by-component approach is used to construct point sets $\mathcal{P}_{N,s}$ consisting of N points in the s-dimensional unit cube satisfying

$$D_N^*(\mathcal{P}_{N,s}) = O\left(\frac{s^{3/2}}{N^{1/2}} \left(\log\left(\frac{N}{s} + 1 \right) \right)^{1/2} \right)$$

in run-time $O(c^s N^{(s+3)/2} \left(\log \frac{N}{s}\right)^{-(s+1)/2} s^{1/4-s/2})$, where $c > 0$ is a constant and where the implied constant in the O-notation is independent of s and N. The improved run-time has to be payed with a worse dependence of the bound for the star discrepancy on the dimension s. Nevertheless, numerical tests of the component-by-component algorithm in [58] suggest that the star discrepancy only grows linearly in s rather than with $s^{3/2}$.

An overview of many open questions concerning this topic can be found in [91, 200]. An effective construction of point sets whose star discrepancy satisfies a bound like in Theorem 3.54 is still not known. An answer to this question would be certainly a major contribution, especially for users of QMC rules.

Let us turn now our attention to this problem but for the L_2-discrepancy instead of star discrepancy. Similarly as for the star discrepancy we define the following quantity.

Definition 3.56 For integers $s, N \in \mathbb{N}$ let

$$\mathrm{disc}_2(N, s) = \inf_{\mathcal{P}} L_{2,N}(\mathcal{P}),$$

where the infimum is extended over all point sets $\mathcal{P}$ consisting of N points in the s-dimensional unit cube. Then $\mathrm{disc}_2(N, s)$ is called the *Nth minimal L_2-discrepancy*.

In contrary to the star discrepancy here it makes little sense to ask for the smallest cardinality of a point set with L_2-discrepancy of at most some $\varepsilon > 0$. The reason for this is that the L_2-discrepancy of the empty point set in the s-dimensional unit cube is exactly $3^{-s/2}$, which follows from Proposition 2.15, or in other words, $\mathrm{disc}_2(0, s) = 3^{-s/2}$. Thus for s large enough, the empty set has always L_2-discrepancy smaller than ε. (This is not the case for the star discrepancy which is always one for the empty set.) This may suggest that for large s, the L_2-discrepancy is not properly scaled.

We define the following quantity.

Definition 3.57 For $\varepsilon > 0$ we define

$$N_2(s, \varepsilon) = \min\left\{N \in \mathbb{N} \,:\, \mathrm{disc}_2(N, s) \le \varepsilon\, \mathrm{disc}_2(0, s)\right\},$$

the so-called *inverse* of L_2-discrepancy.

Here the situation is quite different. The inverse of L_2-discrepancy depends at least exponentially on the dimension s. This was shown in [247, 264] in a much more general setting. In Complexity Theory this exponential dependence on the dimension is called *intractability* or the *curse of dimensionality*.

Proposition 3.58 *For $\varepsilon \in (0, 1)$ we have*

$$N_2(s, \varepsilon) \ge (1 - \varepsilon^2)\left(\frac{9}{8}\right)^s.$$

Proof Proposition 2.15 states that for any point set $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$

in $[0,1)^s$ we have

$$(L_{2,N}(\mathcal{P}))^2 = \frac{1}{3^s} - \frac{2}{N}\sum_{n=0}^{N-1}\prod_{i=1}^{s}\frac{1-x_{n,i}^2}{2} + \frac{1}{N^2}\sum_{m,n=0}^{N-1}\prod_{i=1}^{s}\min(1-x_{m,i},1-x_{n,i}),$$

where $x_{n,i}$ is the ith component of the point $\boldsymbol{x}_n$.

With

$$\kappa_s := \sup_{\boldsymbol{x}\in[0,1]^s} 3^{s/2}\prod_{i=1}^{s}\frac{1-x_j^2}{2\sqrt{1-x_i}} \le \left(\frac{8}{9}\right)^{s/2}$$

(note that the function $x \mapsto (1-x^2)/\sqrt{1-x}$ for $x \in [0,1]$ attains its maximum at $x = 1/3$) we obtain $\prod_{i=1}^{s}\frac{1-x_{n,i}^2}{2} \le \frac{\kappa_s}{3^{s/2}}\prod_{i=1}^{s}\sqrt{1-x_{n,i}}$ and hence

$$\begin{aligned}\frac{1}{N}\sum_{n=0}^{N-1}\prod_{i=1}^{s}\frac{1-x_{n,i}^2}{2} &\le \frac{\kappa_s}{3^{s/2}}\frac{1}{N}\sum_{n=0}^{N-1}\prod_{i=1}^{s}\sqrt{1-x_{n,i}}\\ &\le \frac{\kappa_s}{3^{s/2}}\sqrt{\frac{1}{N}\sum_{n=0}^{N-1}\prod_{i=1}^{s}(1-x_{n,i})},\end{aligned}$$

where we used Cauchy-Schwarz' inequality for the second estimate.

On the other hand we have

$$\frac{1}{N^2}\sum_{m,n=0}^{N-1}\prod_{i=1}^{s}\min(1-x_{m,i},1-x_{n,i}) \ge \frac{1}{N^2}\sum_{n=0}^{N-1}\prod_{i=1}^{s}(1-x_{n,i})\,.$$

Letting $y := \left(\frac{1}{N}\sum_{n=0}^{N-1}\prod_{i=1}^{s}(1-x_{n,i})\right)^{1/2}$ we therefore obtain

$$(L_{2,N}(\mathcal{P}))^2 \ge \frac{1}{3^s} - \frac{2\kappa_s}{3^{s/2}}y + \frac{y^2}{N}.$$

The last term becomes minimal for $y = N\kappa_s 3^{-s/2}$ and hence,

$$(L_{2,N}(\mathcal{P}))^2 \ge \frac{1}{3^s}\left(1 - N\kappa_s^2\right) \ge \frac{1}{3^s}\left(1 - N\left(\frac{8}{9}\right)^s\right).$$

If we assume now that $L_{2,N}(\mathcal{P}) \le \varepsilon\cdot 3^{-s/2}$, then it follows that $\varepsilon^2 \ge 1 - N\,(8/9)^s$ and hence

$$N \ge \left(1-\varepsilon^2\right)\left(\frac{9}{8}\right)^s. \qquad \square$$

For a more detailed discussion of tractability of various notions of discrepancy we refer to the work of Novak & Woźniakowski [197, 198, 199, 200].

3.6 Weighted discrepancy

Apart from the classical concept of discrepancy there is also the idea of weighted discrepancy as introduced by Sloan & Woźniakowski [247], who observed that different coordinates may have different influence on the quality of approximation of an integral by a QMC rule.

We assume that we are given nonnegative real numbers $\gamma_{\mathfrak{u},s}$ for $\mathfrak{u} \subseteq \mathcal{I}_s$, the so-called *weights* corresponding to the projection on the coordinates whose indices are in $\mathfrak{u}$. We collect these weights in the set $\boldsymbol{\gamma} = \{\gamma_{\mathfrak{u},s} : \mathfrak{u} \subseteq \mathcal{I}_s\}$.

Definition 3.59 For a point set $\mathcal{P}$ consisting of N points in the s-dimensional unit cube and given weights $\boldsymbol{\gamma}$, the weighted star discrepancy $D^*_{N,\boldsymbol{\gamma}}$ is given by

$$D^*_{N,\boldsymbol{\gamma}}(\mathcal{P}) = \sup_{\boldsymbol{z}\in(0,1]^s} \max_{\emptyset\neq\mathfrak{u}\subseteq\mathcal{I}_s} \gamma_{\mathfrak{u},s}|\Delta_{\mathcal{P}}((\boldsymbol{z}_{\mathfrak{u}},1))|.$$

For $1 \le q < \infty$, the weighted L_q-discrepancy $L_{q,N,\boldsymbol{\gamma}}$ of $\mathcal{P}$ is given by

$$L_{q,N,\boldsymbol{\gamma}}(\mathcal{P}) = \left(\sum_{\emptyset\neq\mathfrak{u}\subseteq\mathcal{I}_s} \gamma_{\mathfrak{u},s} \int_{[0,1]^{|\mathfrak{u}|}} |\Delta_{\mathcal{P}}((\boldsymbol{z}_{\mathfrak{u}},1))|^q \,\mathrm{d}\boldsymbol{z}_u\right)^{1/q}.$$

Here $\Delta_{\mathcal{P}}$ is the discrepancy function of $\mathcal{P}$ as defined in Definition 2.13.

In the literature, mainly the following kind of weights are studied:

- *Product weights* which are weights of the form $\gamma_{\mathfrak{u},s} = \prod_{i\in\mathfrak{u}} \gamma_{i,s}$, for $\emptyset \neq \mathfrak{u} \subseteq \mathcal{I}_s$, where $\gamma_{i,s}$ is the weight associated with the ith component. In this case we simply write $\boldsymbol{\gamma} = (\gamma_{i,s})_{i=1}^s$. Often the weights $\gamma_{i,s}$ have no dependence on s, i.e., $\gamma_{i,s} = \gamma_i$.
- *Finite-order weights* of fixed order $k \in \mathbb{N}$ which are weights with $\gamma_{\mathfrak{u},s} = 0$ for all $\mathfrak{u} \subseteq \mathcal{I}_s$ with $|\mathfrak{u}| > k$.

Within this book we restrict ourselves mainly to the case of product weights.

If it is not important, we suppress a possible dependence of the weights on the dimension s in the following and we simply write $\gamma_{\mathfrak{u}}$ instead of $\gamma_{\mathfrak{u},s}$.

Note that for $\gamma_{\mathcal{I}_s,s} = 1$ and $\gamma_{\mathfrak{u},s} = 0$ for all $\mathfrak{u} \subsetneq \mathcal{I}_s$ we obtain the usual definitions of L_q- or star discrepancy. Hence Definition 3.59 is a generalisation of Definition 2.14 and Definition 3.19, respectively. Furthermore, in the case of product weights, we also have $D^*_{\mathbf{1},N} = D^*_N$ when $\mathbf{1} = (1)_{i\ge1}$, the sequence of weights where every weight is equal to one.

The two most important cases for weighted discrepancies are those of the weighted L_2-discrepancy and the weighted star discrepancy. Many results for the classical definitions can easily be generalised to results for the

weighted discrepancies. For example, also here we have a compact formula for the evaluation of the weighted L_2-discrepancy of a finite point set (see Proposition 2.15 for the unweighted case).

Proposition 3.60 *For any point set $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$ in $[0,1)^s$ we have*

$$(L_{2,N,\boldsymbol{\gamma}}(\mathcal{P}))^2 = \sum_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{I}_s} \gamma_{\mathfrak{u},s} \times \left[\frac{1}{3^{|\mathfrak{u}|}} - \frac{2}{N} \sum_{n=0}^{N-1} \prod_{i \in \mathfrak{u}} \frac{1 - x_{n,i}^2}{2} + \frac{1}{N^2} \sum_{m,n=0}^{N-1} \prod_{i \in \mathfrak{u}} \min\left(1 - x_{m,i}, 1 - x_{n,i}\right) \right],$$

where $x_{n,i}$ is the ith component of the point $\boldsymbol{x}_n$.

For the weighted star discrepancy we have the following generalisation of Theorem 3.28.

Theorem 3.61 *Let $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$ be a point set in $[0,1)^s$ with $\boldsymbol{x}_n$ of the form $\boldsymbol{x}_n = \{\boldsymbol{y}_n / b^m\}$ with $\boldsymbol{y}_n \in \mathbb{Z}^s$ and integers $m \geq 1$ and $b \geq 2$. Then we have*

$$D^*_{N,\boldsymbol{\gamma}}(\mathcal{P}) \leq \max_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{I}_s} \gamma_{\mathfrak{u},s} \left(1 - \left(1 - \frac{1}{b^m}\right)^{|\mathfrak{u}|}\right) + \max_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{I}_s} \gamma_{\mathfrak{u},s} \sum_{\substack{\boldsymbol{k} \in \mathbb{N}_0^{|\mathfrak{u}|} \\ 0 < |\boldsymbol{k}|_\infty < b^m}} \rho_b(\boldsymbol{k}) \left| \frac{1}{N} \sum_{n=0}^{N-1} {}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_{n,\mathfrak{u}}) \right|,$$

where $\boldsymbol{x}_{n,\mathfrak{u}}$ is the projection of $\boldsymbol{x}_n$ to the coordinates given by $\mathfrak{u}$.

Proof We have

$$D^*_{N,\boldsymbol{\gamma}}(\mathcal{P}) = \sup_{\boldsymbol{z} \in (0,1]^s} \max_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{I}_s} \gamma_{\mathfrak{u},s} |\Delta_{\mathcal{P}}((\boldsymbol{z}_{\mathfrak{u}}, 1))| \leq \max_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{I}_s} \gamma_{\mathfrak{u},s} D^*_N(\mathcal{P}_{\mathfrak{u}}),$$

where $\mathcal{P}_{\mathfrak{u}} = \{\boldsymbol{x}_{0,\mathfrak{u}}, \ldots, \boldsymbol{x}_{N-1,\mathfrak{u}}\}$ in $[0,1)^{|\mathfrak{u}|}$ consists of the points of $\mathcal{P}$ projected to the components whose indices are in $\mathfrak{u}$. For any $\emptyset \neq \mathfrak{u} \subseteq \mathcal{I}_s$ we have from Theorem 3.28 that

$$D^*_N(\mathcal{P}_{\mathfrak{u}}) \leq 1 - \left(1 - \frac{1}{b^m}\right)^{|\mathfrak{u}|} + \sum_{\substack{\boldsymbol{k} \in \mathbb{N}_0^{|\mathfrak{u}|} \\ 0 < |\boldsymbol{k}|_\infty < b^m}} \rho_b(\boldsymbol{k}) \left| \frac{1}{N} \sum_{n=0}^{N-1} {}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_{n,\mathfrak{u}}) \right|,$$

and the result follows. □

One of the reasons for introducing a weighted L_2-discrepancy is that with this concept one can overcome the curse of dimensionality for the L_2-discrepancy under suitable conditions on the weights $\boldsymbol{\gamma}$. And also for the weighted star discrepancy one can obtain a weaker dependence on the dimension for suitable choices of weights.

Definition 3.62 For integers $s, N \in \mathbb{N}$ let

$$\mathrm{disc}^*_{\boldsymbol{\gamma}}(N,s) = \inf_{\mathcal{P}} D^*_{N,\boldsymbol{\gamma}}(\mathcal{P}),$$

and let

$$\mathrm{disc}_{2,\boldsymbol{\gamma}}(N,s) = \inf_{\mathcal{P}} L_{2,N,\boldsymbol{\gamma}}(\mathcal{P}),$$

where the infimum is in both cases extended over all point sets $\mathcal{P}$ consisting of N points in the s-dimensional unit cube. Then $\mathrm{disc}^*_{\boldsymbol{\gamma}}(N,s)$ is called the *Nth minimal weighted star discrepancy* and $\mathrm{disc}_{2,\boldsymbol{\gamma}}(N,s)$ is called the *Nth minimal weighted L_2-discrepancy*. Furthermore, for $\varepsilon > 0$ define

$$N^*_{\boldsymbol{\gamma}}(s,\varepsilon) = \min\{N \in \mathbb{N} : \mathrm{disc}^*_{\boldsymbol{\gamma}}(N,s) \le \varepsilon\}$$

and

$$N_{2,\boldsymbol{\gamma}}(s,\varepsilon) = \min\{N \in \mathbb{N} : \mathrm{disc}_{2,\boldsymbol{\gamma}}(N,s) \le \varepsilon \cdot \mathrm{disc}_{2,\boldsymbol{\gamma}}(0,s)\},$$

the *inverse* of weighted star and weighted L_2-discrepancy, respectively.

Definition 3.63 We say that the weighted star discrepancy and the weighted L_2-discrepancy, respectively, is *polynomial tractable*, if there exist nonnegative C, α and β such that

$$N^*_{\boldsymbol{\gamma}}(s,\varepsilon) \le C s^\alpha \varepsilon^{-\beta}, \quad \text{and} \quad N_{2,\boldsymbol{\gamma}}(s,\varepsilon) \le C s^\alpha \varepsilon^{-\beta},$$

respectively holds for all dimensions $s \in \mathbb{N}$ and for all $\varepsilon \in (0,1)$. This behaviour is also called *tractability*. The infima of α and β such that such an inequality holds are called the *s-exponent* and the *ε-exponent* of tractability. We say that the weighted star and the weighted L_2-discrepancy, respectively, is *strongly tractable*, if the above inequality holds with $\alpha = 0$. In this context one also speaks of *strong tractability*.

We consider the case of the weighted L_2-discrepancy first.

Theorem 3.64 *Assume that the weights $\boldsymbol{\gamma}$ are such that*

$$B_{\boldsymbol{\gamma}} := \sup_{s\in\mathbb{N}} \frac{\sum_{\emptyset\neq \mathrm{u}\subseteq\mathcal{I}_s} \gamma_{\mathrm{u},s}\left(\frac{1}{2^{|\mathrm{u}|}} - \frac{1}{3^{|\mathrm{u}|}}\right)}{\sum_{\emptyset\neq \mathrm{u}\subseteq\mathcal{I}_s} \gamma_{\mathrm{u},s}\frac{1}{3^{|\mathrm{u}|}}} < \infty,$$

then the weighted L_2-discrepancy is strongly tractable and the ε-exponent is

at most two. Moreover, in the case of product weights (independent of s) with decreasing weights, the weighted L_2-discrepancy is strongly tractable, if and only if $\sum_{i=1}^{\infty} \gamma_i < \infty$.

Proof Averaging the squared weighted L_2-discrepancy over all $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_N$ from the s-dimensional unit cube yields

$$\int_{[0,1]^{sN}} \left(L_{2,N,\boldsymbol{\gamma}}(\{\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_N\})\right)^2 \,\mathrm{d}\boldsymbol{\tau}_1 \cdots \mathrm{d}\boldsymbol{\tau}_N = \frac{1}{N} \sum_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{I}_s} \gamma_{\mathfrak{u},s} \left(\frac{1}{2^{|\mathfrak{u}|}} - \frac{1}{3^{|\mathfrak{u}|}}\right).$$

Hence there exists a point set $\mathcal{P}$ consisting of N points in the s-dimensional unit cube such that

$$L_{2,N,\boldsymbol{\gamma}}(\mathcal{P}) \leq \frac{1}{\sqrt{N}} \left(\sum_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{I}_s} \gamma_{\mathfrak{u},s} \left(\frac{1}{2^{|\mathfrak{u}|}} - \frac{1}{3^{|\mathfrak{u}|}}\right)\right)^{1/2} \leq \frac{\sqrt{B_{\boldsymbol{\gamma}}}}{\sqrt{N}} \left(\sum_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{I}_s} \frac{\gamma_{\mathfrak{u},s}}{3^{|\mathfrak{u}|}}\right)^{1/2}.$$

The last term is smaller than $\varepsilon \left(\sum_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{I}_s} \frac{\gamma_{\mathfrak{u},s}}{3^{|\mathfrak{u}|}}\right)^{1/2}$ if $N \geq B_{\boldsymbol{\gamma}} \varepsilon^{-2}$. This means that $N_{2,\boldsymbol{\gamma}}(s, \varepsilon) \leq \lceil B_{\boldsymbol{\gamma}} \varepsilon^{-2} \rceil$ and hence we have strong tractability with ε-exponent of at most two.

Assume that we are given product weights which are independent of the dimension s, i.e. $\gamma_{\mathfrak{u},s} = \prod_{i \in \mathfrak{u}} \gamma_i$ with a sequence $\gamma_1, \gamma_2, \ldots \geq 0$. In this case we have

$$\begin{aligned}
\frac{\sum_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{I}_s} \gamma_{\mathfrak{u},s} \left(\frac{1}{2^{|\mathfrak{u}|}} - \frac{1}{3^{|\mathfrak{u}|}}\right)}{\sum_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{I}_s} \gamma_{\mathfrak{u},s} \frac{1}{3^{|\mathfrak{u}|}}} &= \frac{\prod_{i=1}^{s} \left(1 + \frac{\gamma_i}{2}\right) - \prod_{i=1}^{s} \left(1 + \frac{\gamma_i}{3}\right)}{\prod_{i=1}^{s} \left(1 + \frac{\gamma_i}{3}\right)} \\
&= \frac{\prod_{i=1}^{s} \left(1 + \frac{\gamma_i}{2}\right)}{\prod_{i=1}^{s} \left(1 + \frac{\gamma_i}{3}\right)} - 1 \leq \prod_{i=1}^{s} \left(1 + \frac{\gamma_i}{6}\right) \\
&\leq \mathrm{e}^{\sum_{i=1}^{s} \log(1+\gamma_i/6)} \leq \mathrm{e}^{(\sum_{i=1}^{s} \gamma_i)/6}.
\end{aligned}$$

Hence $B_{\boldsymbol{\gamma}} < \infty$ if $\sum_{i=1}^{\infty} \gamma_i < \infty$ and we obtain strong tractability.

On the other hand, using the lower bound on the unweighted L_2-discrepancy from the proof of Proposition 3.58 we have

$$\begin{aligned}
\left(L_{2,N,\boldsymbol{\gamma}}(\mathcal{P})\right)^2 &\geq \sum_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{I}_s} \gamma_{\mathfrak{u},s} \frac{1}{3^{|\mathfrak{u}|}} \left(1 - N \left(\frac{8}{9}\right)^{|\mathfrak{u}|}\right) \\
&= -1 + \prod_{i=1}^{s} \left(1 + \frac{\gamma_i}{3}\right) + N - N \prod_{i=1}^{s} \left(1 + \frac{8\gamma_i}{27}\right).
\end{aligned}$$

Assume we had strong tractability, i.e., there exist nonnegative C and β

with the property that $N_{2,\boldsymbol{\gamma}}(s,\varepsilon) \le C\varepsilon^{-\beta}$ for all $s \in \mathbb{N}$ and all $\varepsilon > 0$. Then for $N = N_{2,\boldsymbol{\gamma}}(s,\varepsilon)$ we have

$$\begin{aligned}\varepsilon^2 \prod_{i=1}^{s}\left(1+\frac{\gamma_i}{3}\right) &\ge -1 + \prod_{i=1}^{s}\left(1+\frac{\gamma_i}{3}\right) + N - N\prod_{i=1}^{s}\left(1+\frac{8\gamma_i}{27}\right)\\ &\ge \prod_{i=1}^{s}\left(1+\frac{\gamma_i}{3}\right) - N\prod_{i=1}^{s}\left(1+\frac{8\gamma_i}{27}\right).\end{aligned}$$

Hence, for $0 < \varepsilon < 1$, we have

$$C\varepsilon^{-\beta} \ge N \ge (1-\varepsilon^2)\prod_{i=1}^{s}\frac{1+\frac{\gamma_i}{3}}{1+\frac{8\gamma_i}{27}} = (1-\varepsilon^2)\prod_{i=1}^{s}\left(1+\frac{\gamma_i}{27+8\gamma_i}\right).$$

Obviously, the sequence $(\gamma_i)_{i\ge1}$ must be bounded since otherwise we would have $\frac{\gamma_i}{27+8\gamma_i} \ge \frac{1}{16}$ for infinitely many $i \in \mathbb{N}$ and hence $C\varepsilon^{-\beta} \ge (1-\varepsilon^2)\left(1+\frac{1}{16}\right)^d$ for infinitely many $d \in \mathbb{N}$ which is certainly a contradiction.

For bounded γ_i's, say $\gamma_i \le M$ for all $i \in \mathbb{N}$ we obtain

$$C\varepsilon^{-\beta} \ge (1-\varepsilon^2)\prod_{i=1}^{s}\left(1+\frac{1}{27+8M}\gamma_i\right) \ge (1-\varepsilon^2)\frac{1}{27+8M}\sum_{i=1}^{s}\gamma_i$$

and thus we must have $\sum_{i=1}^{\infty}\gamma_i < \infty$. □

For the star discrepancy we have tractability already for the unweighted case (with s-exponent of at most one and ε-exponent of at most two). From this it follows immediately that the weighted star discrepancy is also tractable with s-exponent of at most one and ε-exponent of at most two as long as the weights are bounded. However, under a very mild condition on the weights one can even obtain tractability with s-exponent equal to zero. The following result was first proved in [108].

Theorem 3.65 *If*

$$C_{\boldsymbol{\gamma}} := \sup_{s=1,2,\ldots}\max_{\emptyset\ne\mathfrak{u}\subseteq\mathcal{I}_s}\gamma_{\mathfrak{u},s}\sqrt{|\mathfrak{u}|} < \infty, \tag{3.23}$$

then for all $N, s \in \mathbb{N}$ we have

$$\mathrm{disc}^*_{\boldsymbol{\gamma}}(N,s) \le \frac{2\sqrt{2}C_{\boldsymbol{\gamma}}}{\sqrt{N}}\left(\log\left(\left\lceil\rho_s\sqrt{N}\right\rceil+1\right)+\log\left(2(\mathrm{e}-1)s\right)\right)^{1/2}, \tag{3.24}$$

where $\rho_s = \frac{s}{2(\log 2)^{1/2}}$. Hence for any $0 < \delta < 1$ there exists a $c_\delta > 0$ such that

$$N^*_{\boldsymbol{\gamma}}(s,\varepsilon) \le \left\lceil c_\delta \varepsilon^{-2/(1-\delta)}(\log s+1)^{1/(1-\delta)}\right\rceil, \tag{3.25}$$

i.e., the weighted star discrepancy is tractable with s-exponent equal to zero and with ε-exponent at most two.

We stress that we do not have strong tractability in this case as we still have the logarithmic dependence on the dimension s.

Note that condition (3.23) is a very mild condition on the weights. For example for bounded finite order weights it is always fulfilled. In the case of product weights (independent of s) it is enough that the weights γ_j are decreasing and that $\gamma_j < 1$ for an index $j \in \mathbb{N}$. In fact, we have

$$\max_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{I}_s} \gamma_{\mathfrak{u},s} \sqrt{|\mathfrak{u}|} = \max_{u=1,\ldots,s} \sqrt{u} \prod_{i=1}^{u} \gamma_i$$

and hence $C_{\boldsymbol{\gamma}} = \sup_{s=1,2,\ldots} \sqrt{s} \prod_{i=1}^{s} \gamma_i$. We have

$$\frac{\sqrt{s} \prod_{i=1}^{s} \gamma_i}{\sqrt{s+1} \prod_{i=1}^{s+1} \gamma_i} = \sqrt{\frac{s}{s+1}} \frac{1}{\gamma_{s+1}} > 1$$

for s large enough and therefore it follows that $C_{\boldsymbol{\gamma}} < \infty$. For example, if $\gamma_i = 1/\log(i+1)$, then $C_{\boldsymbol{\gamma}} = \frac{\sqrt{2}}{\log 2 \log 3}$.

Proof of Theorem 3.65 For given number of points N and dimension s and $0 < c_{\mathfrak{u}} \leq 1$ for all $\emptyset \neq \mathfrak{u} \subseteq \mathcal{I}_s$ we consider the set

$$A := \Bigg\{ \mathcal{P}_{N,s} \subseteq [0,1)^s \,:\, |\mathcal{P}_{N,s}| = N \text{ and for all } \emptyset \neq \mathfrak{u} \subseteq \mathcal{I}_s$$
$$D_N^*(\mathcal{P}_{N,s,\mathfrak{u}}) \leq \frac{2\sqrt{2}}{\sqrt{N}} \left(|\mathfrak{u}| \log\left(\left\lceil \rho_{|\mathfrak{u}|} \sqrt{N} \right\rceil + 1 \right) + \log\left(\frac{2}{c_{\mathfrak{u}}} \right) \right)^{1/2} \Bigg\},$$

where $\mathcal{P}_{N,s,\mathfrak{u}} := \{\boldsymbol{x}_{0,\mathfrak{u}}, \ldots, \boldsymbol{x}_{N-1,\mathfrak{u}}\}$ if $\mathcal{P}_{N,s} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$ and where $\rho_{|\mathfrak{u}|} = \frac{|\mathfrak{u}|}{2(\log 2)^{1/2}}$. Furthermore, for $\emptyset \neq \mathfrak{u} \subseteq \mathcal{I}_s$, we define

$$A_{\mathfrak{u}} = A_{\mathfrak{u}}(c_{\mathfrak{u}})$$
$$:= \Bigg\{ \mathcal{P}_{N,s} \subseteq [0,1)^s \,:\, |\mathcal{P}_{N,s}| = N \text{ and}$$
$$D_N^*(\mathcal{P}_{N,s,\mathfrak{u}}) \leq \frac{2\sqrt{2}}{\sqrt{N}} \left(|\mathfrak{u}| \log\left(\left\lceil \rho_{|\mathfrak{u}|} \sqrt{N} \right\rceil + 1 \right) + \log\left(\frac{2}{c_{\mathfrak{u}}} \right) \right)^{1/2} \Bigg\}.$$

From Remark 3.55 we know that $\mathrm{Prob}[A_{\mathfrak{u}}(c_{\mathfrak{u}})] > 1 - c_{\mathfrak{u}}$. Then we have

$A = \bigcap_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{I}_s} A_{\mathfrak{u}}$ and hence

$$\begin{aligned}\mathrm{Prob}[A] &= \mathrm{Prob}\left[\bigcap_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{I}_s} A_{\mathfrak{u}}\right] = 1 - \mathrm{Prob}\left[\bigcup_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{I}_s} A_{\mathfrak{u}}^c\right] \\ &\geq 1 - \sum_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{I}_s} \mathrm{Prob}\left[A_{\mathfrak{u}}^c\right] \geq 1 - \sum_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{I}_s} c_{\mathfrak{u}}.\end{aligned}$$

If we choose $c_{\mathfrak{u}} := cs^{-|\mathfrak{u}|}$ with a constant $0 < c \leq (\mathrm{e}-1)^{-1}$, then we obtain

$$\mathrm{Prob}[A] \geq 1 - \sum_{u=1}^{s} \binom{s}{u} \frac{c}{s^u} = 1 + c - c\left(1 + \frac{1}{s}\right)^s > 1 + c - c \cdot \mathrm{e} \geq 0.$$

Thus we have shown that there exists a point set $\mathcal{P}_{N,s} \subseteq [0,1)^s$ such that for each $\emptyset \neq \mathfrak{u} \subseteq \mathcal{I}_s$ we have

$$\begin{aligned}D_N^*(\mathcal{P}_{N,s,\mathfrak{u}}) &\leq \frac{2\sqrt{2}}{\sqrt{N}} \left(|\mathfrak{u}| \log\left(\left\lceil \rho_{|\mathfrak{u}|}\sqrt{N}\right\rceil + 1\right) + \log\left(2(\mathrm{e}-1)s^{|\mathfrak{u}|}\right)\right)^{1/2} \\ &\leq \frac{2\sqrt{2}\sqrt{|\mathfrak{u}|}}{\sqrt{N}} \left(\log\left(\left\lceil \rho_{|\mathfrak{u}|}\sqrt{N}\right\rceil + 1\right) + \log\left(2(\mathrm{e}-1)s\right)\right)^{1/2}.\end{aligned}$$

For the weighted star discrepancy of this point set we obtain

$$\begin{aligned}&D_{N,\boldsymbol{\gamma}}^*(\mathcal{P}_{N,s}) \\ &\leq \frac{2\sqrt{2}}{\sqrt{N}} \left(\log\left(\left\lceil \rho_{|\mathfrak{u}|}\sqrt{N}\right\rceil + 1\right) + \log\left(2(\mathrm{e}-1)s\right)\right)^{1/2} \max_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{I}_s} \gamma_{\mathfrak{u},s}\sqrt{|\mathfrak{u}|}.\end{aligned}$$

Assume now that $C_{\boldsymbol{\gamma}} := \sup_{s=1,2,\ldots} \max_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{I}_s} \gamma_{\mathfrak{u},s}\sqrt{|\mathfrak{u}|} < \infty$. Then we obtain

$$D_{N,\boldsymbol{\gamma}}^*(\mathcal{P}_{N,s}) \leq \frac{2\sqrt{2}C_{\boldsymbol{\gamma}}}{\sqrt{N}} \left(\log\left(\left\lceil \rho_{|\mathfrak{u}|}\sqrt{N}\right\rceil + 1\right) + \log\left(2(\mathrm{e}-1)s\right)\right)^{1/2}$$

and (3.24) follows.

For any $\delta > 0$ there exists a $c_\delta > 0$ such that

$$C_{\boldsymbol{\gamma}} 2\sqrt{2} \left(\log\left(\left\lceil \rho_s\sqrt{N}\right\rceil + 1\right) + \log\left(2(\mathrm{e}-1)s\right)\right)^{1/2} \leq \left(c_\delta N^\delta (\log s + 1)\right)^{1/2}.$$

Hence it follows from (3.24) that $N \geq c_\delta \varepsilon^{-2/(1-\delta)} (\log s + 1)^{1/(1-\delta)}$ implies $\mathrm{disc}_{\boldsymbol{\gamma}}^*(N,s) \leq \varepsilon$ and therefore

$$N_{\boldsymbol{\gamma}}^*(s,\varepsilon) \leq \left\lceil c_\delta \varepsilon^{-2/(1-\delta)} (\log s + 1)^{1/(1-\delta)} \right\rceil. \qquad \square$$

We close this section by showing that the logarithmic factor in the dimension in the tractability result from Theorem 3.65 is indeed necessary for a large class of weights. This implies that the star discrepancy is not strongly tractable for such weights. In particular, this includes finite order weights of order $k \geq 2$ if all the weights of order 2 are bounded below by a constant $c > 0$.

To prove this lower bound we need an elementary lemma. For $\mathfrak{u} \subseteq \mathcal{I}_s$ and $k \in \{0,1\}$ let

$$B_k(\mathfrak{u}) = \left\{ \boldsymbol{x} = (x_1, \ldots, x_s) \in [0,1)^s \,:\, x_i \in \left[\frac{k}{2}, \frac{k+1}{2}\right) \text{ for } i \in \mathfrak{u} \right\}.$$

Lemma 3.66 *Let $\mathcal{P}_{N,s} \subseteq [0,1)^s$ with $|\mathcal{P}_{N,s}| = N$. Then there exists $\mathfrak{u} \subseteq \mathcal{I}_s$ with cardinality at least $s/2^N$ such that one of the sets $B_0(\mathfrak{u})$ and $B_1(\mathfrak{u})$ contains at least half of the points of $\mathcal{P}_{N,s}$.*

Proof There exists $\mathfrak{u}_0 \subseteq \mathcal{I}_s$ with cardinality at least $s/2$ and $k_0 \in \{0,1\}$ such that $\boldsymbol{x}_0 \in B_{k_0}(\mathfrak{u}_0)$. Inductively, for $1 \leq h < N$, we can choose $\mathfrak{u}_h \subseteq \mathfrak{u}_{h-1}$ with cardinality at least $s/2^{h+1}$ and $k_h \in \{0,1\}$ such that $\boldsymbol{x}_h \in B_{k_h}(\mathfrak{u})$. Set $\mathfrak{u} = \mathfrak{u}_{N-1}$ and let $k \in \{0,1\}$ be such that at least half of the k_h, $0 \leq h < N$ are equal to k. Then the cardinality of $\mathfrak{u}$ is at least $s/2^N$ and at least half of the points $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}$ are in $B_k(\mathfrak{u})$. □

Now we give the announced lower bound for the weighted star discrepancy which was first proved in [108].

Theorem 3.67 *If the weights $\boldsymbol{\gamma} = \{\gamma_{\mathfrak{u},s} \,:\, \mathfrak{u} \subseteq \mathcal{I}_s\}$ are such that there exists a constant $c > 0$ with $\gamma_{\mathfrak{u},s} \geq c$ for all $\mathfrak{u} \subseteq \mathcal{I}_s$ with cardinality two, then for all $N, s \in \mathbb{N}$ with $s \geq 2^{N+1}$ we have*

$$\mathrm{disc}^*_{\boldsymbol{\gamma}}(N,s) \geq \frac{c}{12}.$$

In particular, the weighted star discrepancy is not strongly tractable for such weights.

Proof Let $\mathcal{P}$ be a point set consisting of N points in the s-dimensional unit cube where $s \geq 2^{N+1}$. With Lemma 3.66 we find $\mathfrak{u}_0 \subseteq \mathcal{I}_s$ with cardinality 2 such that one of the sets $B_0(\mathfrak{u}_0)$ or $B_1(\mathfrak{u}_0)$ contains at least $N/2$ points of $\mathcal{P}$. Without loss of generality we assume that $\mathfrak{u}_0 = \{1,2\}$. Let $\boldsymbol{z}^{(0)} = (1/2, 1/2, 1/2, \ldots, 1/2)$, $\boldsymbol{z}^{(1)} = (1, 1/2, 1/2, \ldots, 1/2)$ and $\boldsymbol{z}^{(2)} = (1/2, 1, 1/2, \ldots, 1/2)$. Furthermore, let n_0, n_1, n_2 be the number of points in the point set $\mathcal{P}$ which are contained in the boxes $I_1 \times I_2 \times [0,1)^{s-2}$ for $I_1 = I_2 = [0, 1/2)$, $I_1 = [1/2, 1), I_2 = [0, 1/2)$ and $I_1 = [0, 1/2), I_2 = [1/2, 1)$, respectively.

Let us first assume that the set $B_0(\mathfrak{u}_0)$ contains at least $N/2$ points. Then

$$\Delta_{\mathcal{P}}(\boldsymbol{z}_{\mathfrak{u}_0}^{(0)}, 1) = \frac{A(B_0(\mathfrak{u}_0), N, \mathcal{P})}{N} - \frac{1}{4} \geq \frac{1}{4}$$

which implies

$$D_{N,\boldsymbol{\gamma}}^*(\mathcal{P}) \geq \frac{c}{4}.$$

We now treat the case that the set $B_1(\mathfrak{u}_0)$ contains at least $N/2$ points so that its complement contains at most $N/2$ points, i.e.

$$n_0 + n_1 + n_2 \leq N/2.$$

Then at least one of the following three inequalities holds

$$n_0 + n_1 \leq \frac{5N}{12}, \quad n_0 + n_2 \leq \frac{5N}{12}, \quad n_0 \geq \frac{N}{3}.$$

If the first inequality holds then it follows that

$$\Delta_{\mathcal{P}}(\boldsymbol{z}_{\mathfrak{u}_0}^{(1)}, 1) = \frac{n_0 + n_1}{N} - \frac{1}{2} \leq -\frac{1}{12}.$$

If the second inequality holds, we have

$$\Delta_{\mathcal{P}}(\boldsymbol{z}_{\mathfrak{u}_0}^{(2)}, 1) = \frac{n_0 + n_2}{N} - \frac{1}{2} \leq -\frac{1}{12}.$$

If the third inequality is true then

$$\Delta_{\mathcal{P}}(\boldsymbol{z}_{\mathfrak{u}_0}^{(0)}, 1) = \frac{n_0}{N} - \frac{1}{4} \geq \frac{1}{12}.$$

In any case $D_{N,\boldsymbol{\gamma}}^*(\mathcal{P}) \geq \frac{c}{12}$ and the result follows. □

Again we refer to the work of Novak & Woźniakowski [197, 198, 199, 200] for a more detailed discussion of tractability of various notions of discrepancy.

Exercises

3.1 Show that a uniformly distributed sequence is dense in the unit cube and explain why the converse is not true.

3.2 Which one-dimensional point set $\mathcal{P}$ consisting of N points in $[0, 1)$ minimises $D_N^*(\mathcal{P})$, i.e., for which $\mathcal{P}$ do we have $D_N^*(\mathcal{P}) = \min_{\mathcal{P}'} D_N^*(\mathcal{P}')$, where the minimum is taken over all point sets $\mathcal{P}'$ consisting of N points? Which point set $\mathcal{P}$ consisting of N points minimises $L_{2,N}(\mathcal{P})$? What is the value of $D_N^*(\mathcal{P})$ and $L_{2,N}(\mathcal{P})$ for this point set? *Hint:* Draw the graph of the discrepancy function $\Delta_{\mathcal{P}}$.

3.3 Give a rigorous proof of Theorem 3.3. *Hint:* See [128, Chapter 1, Corollary 1.1].

3.4 Give a proof of Theorem 3.4. *Hint:* See [128, Chapter 1, Corollary 1.2].

3.5 Show that for uniform distribution of the sequence $(\{n\boldsymbol{\alpha}\})_{n\geq 0}$ we necessarily need that $1, \alpha_1, \ldots, \alpha_s$ are linearly independent over $\mathbb{Q}$.

3.6 Show that the b-adic van der Corput sequence is uniformly distributed modulo one just by counting elements of the sequence in intervals, i.e., without the use of Theorem 3.7. *Hint:* Consider elementary b-adic intervals first.

3.7 For $b \geq 2$ the *b-adic diaphony* $F_{b,N}$ (see [86] or [97]) of the first N elements of a sequence $\mathcal{S} = (\boldsymbol{x}_n)_{n\geq 0}$ is defined by

$$F_{b,N}(\mathcal{S}) = \left(\frac{1}{(b+1)^s - 1} \sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^s \\ \boldsymbol{k}\neq\boldsymbol{0}}} \frac{1}{\psi_b(\boldsymbol{k})^2} \left| \frac{1}{N} \sum_{n=0}^{N-1} {}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_n) \right|^2 \right)^{1/2},$$

where for $\boldsymbol{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ it is $\psi_b(\boldsymbol{k}) = \prod_{i=1}^s \psi_b(k_i)$ and for $k \in \mathbb{N}_0$,

$$\psi_b(k) = \begin{cases} 1 & \text{if } k = 0, \\ b^r & \text{if } b^r \leq k < b^{r+1} \text{ where } r \in \mathbb{N}_0. \end{cases}$$

Show that a sequence $\mathcal{S}$ is uniformly distributed modulo one if and only if $\lim_{N\to\infty} F_{b,N}(\mathcal{S}) = 0$ for $b \geq 2$. *Remark:* Compare the b-adic diaphony with the worst-case error for a QMC rule in the Walsh space $\mathscr{H}_{\mathrm{wal},s,b,\alpha,\boldsymbol{\gamma}}$ as given in Exercise 2.15 (especially in the unweighted case and with $\alpha = 2$). *Hint:* See [97, Theorem 3.1].

3.8 For $b \geq 2$ the *b-adic spectral test* $\sigma_{b,N}(\mathcal{S})$ (see [95]) of the first N elements of a sequence $\mathcal{S} = (\boldsymbol{x}_n)_{n\geq 0}$ is defined by

$$\sigma_{b,N}(\mathcal{S}) = \sup_{\substack{\boldsymbol{k}\in\mathbb{N}_0^s \\ \boldsymbol{k}\neq\boldsymbol{0}}} \frac{1}{\psi_b(\boldsymbol{k})} \left| \frac{1}{N} \sum_{n=0}^{N-1} {}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_n) \right|,$$

where ψ_b is defined as in Exercise 3.7. Show that a sequence $\mathcal{S}$ is uniformly distributed modulo one if and only if $\lim_{N\to\infty} \sigma_{b,N}(\mathcal{S}) = 0$ for $b \geq 2$.

3.9 Show that a sequence $\mathcal{S}$ is uniformly distributed modulo one if and only if $\lim_{N\to\infty} D_N(\mathcal{S}) = 0$. *Hint:* See [128, Chapter 2, Theorem 1.1].

3.10 Give a rigorous proof of the right hand inequality in Proposition 3.14 (draw a picture).

3.11 Prove Proposition 3.16. *Hint:* See [128, Chapter 2, Theorem 2.6].

3.12 Let $u_i, v_i, \delta_i \in [0,1]$ be such that $|u_i - v_i| \le \delta_i$ for $1 \le i \le s$. Show that

$$\left| \prod_{i=1}^{s} u_i - \prod_{i=1}^{s} v_i \right| \le 1 - \prod_{i=1}^{s} (1 - \delta_i) \le s \max_{1 \le i \le s} \delta_i.$$

3.13 A finite set $\Gamma \subseteq [0,1]^s$ is a *δ-cover* of $[0,1]^s$ if for every $\boldsymbol{x} = (x_1, \ldots, x_s) \in [0,1]^s$ there exist $\boldsymbol{y}_1 = (y_{1,1}, \ldots, y_{1,s}), \boldsymbol{y}_2 = (y_{2,1}, \ldots, y_{2,s}) \in \Gamma \cup \{\boldsymbol{0}\}$ with $\lambda_s([\boldsymbol{0}, \boldsymbol{y}_2)) - \lambda_s([\boldsymbol{0}, \boldsymbol{y}_1)) \le \delta$ and $y_{1,i} \le x_i \le y_{2,i}$ for all $1 \le i \le s$.

Let Γ be a δ-cover of $[0,1]^s$. Show that then for any N-point set $\mathcal{P} \subseteq [0,1)^s$ we have

$$D_N^*(\mathcal{P}) \le \max_{\boldsymbol{y} \in \Gamma} |\Delta_{\mathcal{P}}(\boldsymbol{y})| + \delta.$$

3.14 Prove a similar formula to that in Proposition 2.15 for the L_q-discrepancy with an even integer q.

3.15 Show that

$$\int_{[0,1]^{sN}} (L_{2,N}(\{\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_N\}))^2 \, \mathrm{d}\boldsymbol{\tau}_1 \cdots \mathrm{d}\boldsymbol{\tau}_N = \frac{1}{N} \left(\frac{1}{2^s} - \frac{1}{3^s} \right).$$

Thus, there exists a point set $\mathcal{P}$ consisting of N points in the s-dimensional unit cube such that

$$L_{2,N}(\mathcal{P}) \le \frac{1}{\sqrt{N}} \left(\frac{1}{2^s} - \frac{1}{3^s} \right)^{1/2}.$$

Hint: Use Proposition 2.15.

3.16 Let $\mathcal{T}_{s,N}(\alpha)$ be the set of all tuples $(\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_N)$ with $\boldsymbol{\tau}_j \in [0,1]^s$ for $1 \le j \le N$ such that

$$L_{2,N}(\{\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_N\}) \le \frac{\alpha}{\sqrt{N}} \left(\frac{1}{2^s} - \frac{1}{3^s} \right)^{1/2}.$$

Use Exercise 3.15 to show that for all $\alpha \ge 1$ we have

$$\lambda_{sN}\left(\mathcal{T}_{s,N}(\alpha)\right) > 1 - \alpha^{-2}.$$

3.17 Use Theorem 3.26 to show that the point set $\{0, 1/N, \ldots, (N-1)/N\}$ in the unit interval has extreme discrepancy of order $1/N$.

3.18 Prove a similar result to that of Theorem 3.28 also for the extreme discrepancy. *Hint:* See [94, Theorem 1].

3.19 For integers $m_1, \ldots, m_s \ge 2$ let

$$\Gamma_{m_1,\ldots,m_s} = \left\{ \left(\frac{n_1}{m_1}, \ldots, \frac{n_s}{m_s} \right) : 0 \le n_i < m_j \text{ for } 1 \le i \le s \right\}$$

be the regular lattice consisting of $N = m_1 \cdots m_s$ points. Show that then

$$D_N^*(\Gamma_{m_1,\ldots,m_s}) = 1 - \prod_{i=1}^{s} \left(1 - \frac{1}{m_i}\right).$$

3.20 Let $\mathcal{P}_s = \left\{\left(\frac{a_1}{4}, \ldots, \frac{a_s}{4}\right) : a_i \in \{1,3\} \text{ for all } 1 \le i \le s\right\}$ be the regular lattice consisting of 2^s points in $[0,1)^s$. Show that $\lim_{s\to\infty} D_{2^s}^*(\mathcal{P}_s) = 1$.

3.21 Let Γ_m be the regular lattice defined by (3.14). Use Theorem 3.27 to show that the extreme discrepancy of Γ_m is given by $D_N(\Gamma_m) = 1 - (1 - 1/m)^s$.

3.22 Show that the star discrepancy of the centred regular lattice Γ_m^{c} consisting of $N = m^s$ points defined by (1.1) is

$$D_N^*(\Gamma_m^{\mathrm{c}}) = 1 - \left(1 - \frac{1}{2m}\right)^s.$$

3.23 In dimensions $s = 1$ and $s = 2$, draw a picture to make the result from Lemma 3.40 more plausible.

3.24 Let $b_1, \ldots, b_s \ge 2$ be pairwise relatively prime integers and let $\mathcal{S}$ be the van der Corput-Halton sequence with bases $b_1, \ldots, b_s$. Show that for any $N \ge 2$ we have

$$D_N(\mathcal{S}) \le \widetilde{c}(b_1, \ldots, b_s)\frac{(\log N)^s}{N} + O\left(\frac{(\log N)^{s-1}}{N}\right),$$

with

$$\widetilde{c}(b_1, \ldots, b_s) = \frac{2^s}{s!}\prod_{i=1}^{s}\frac{\lfloor b_i/2 \rfloor}{\log b_i}.$$

Show further that if $b_1, \ldots, b_s$ are the first s prime numbers, then $\widetilde{c}(b_1, \ldots, b_s) = O(s^{-1})$.

3.25 For integers $s \ge 2$ and $N \ge 2$ consider a generating vector of the form $\boldsymbol{g} = (1, g, g^2, \ldots, g^{s-1}) \in \mathbb{Z}^s$. Such a choice was first proposed by Korobov [122] and therefore such lattice points are often called *Korobov vectors* or *Korobov lattice points*. A lattice point set which is generated by a Korobov vector is often called *Korobov lattice point set*.

Show, by averaging over all $g \in G_N$ that there exists a Korobov vector for which we have

$$R_N((1, g, g^2, \ldots, g^{s-1})) \le \frac{s-1}{N-1}(1 + S_N)^s.$$

Hint: Recall that any nonzero polynomial of degree k over an integral domain has at most k zeros.

3.26 Let $\mathcal{P}$ be a lattice point set consisting of N points in $[0,1)^s$ with generating vector $\boldsymbol{g} \in \mathbb{Z}^s$. Show that the worst-case error for the lattice rule based on $\mathcal{P}$ in the s-dimensional Korobov space $\mathscr{H}_{\mathrm{kor},s,\alpha}$ from Exercise 2.13 is given by

$$e^2(\mathscr{H}_{\mathrm{kor},s,\alpha}, \mathcal{P}) = \sum_{\substack{\boldsymbol{h} \in \mathbb{Z}^s \setminus \{\boldsymbol{0}\} \\ \boldsymbol{g} \cdot \boldsymbol{h} \equiv 0 \pmod N}} \frac{1}{r_\alpha(\boldsymbol{h})},$$

where $r_\alpha(\boldsymbol{h})$ is as in Exercise 2.13.

3.27 Let $e^2_\alpha(\boldsymbol{g}, N)$ be the worst-case integration error for a lattice rule in the s-dimensional Korobov space $\mathscr{H}_{\mathrm{kor},s,\alpha}$ for a lattice point set consisting of N points in $[0,1)^s$ with generating vector $\boldsymbol{g} \in \mathbb{Z}^s$. Let N be a prime. Show that

$$\frac{1}{N^s} \sum_{\boldsymbol{g} \in \{0,\ldots,N-1\}^s} e^2_\alpha(\boldsymbol{g}, N) = \frac{1}{N} \left(-1 + (1 + 2\zeta(\alpha))^s\right).$$

Deduce from this result that for any $1/\alpha < \lambda \le 1$ there exists a generating vector $\boldsymbol{g} \in \{0, \ldots, N-1\}^s$ such that

$$e^2_\alpha(\boldsymbol{g}, N) \le \frac{1}{N^{1/\lambda}} \left(-1 + (1 + 2\zeta(\alpha\lambda))^s\right)^{1/\lambda}.$$

Hint: Use Jensen's inequality which states that for a sequence (a_k) of nonnegative reals and for any $0 < \lambda < 1$ we have $\left(\sum a_k\right)^\lambda \le \sum a_k^\lambda$.

3.28 It has been shown in [57, Theorem 2.3] that there exists a δ-cover Γ of $[0,1]^s$ such that $|\Gamma| \le \left(\left\lceil \frac{s}{s-1} \frac{\log s}{\delta} \right\rceil + 1\right)^d$. Use this result together with Exercise 3.13 to show that

$$\mathrm{disc}^*(N, s) \le \sqrt{2} n^{-1/2} (s \log(\lceil \rho n^{1/2} \rceil + 1) + \log 2),$$

where $\rho = \frac{3 \log 3}{\sqrt{2(3 \log 3 + \log 2)}}$. *Hint:* Follow the proof of Theorem 3.54. *Remark:* This is [57, Theorem 3.2]. Smaller δ-covers as the one from [57, Theorem 2.3] have been constructed in [84].

3.29 Prove Proposition 3.60.

3.30 Prove a similar formula to that of Proposition 3.60 for the weighted L_q-discrepancy with an even integer q. *Hint:* See [154, Theorem 2.1].

3.31 Generalise Proposition 3.16 to the case of weighted star discrepancy.

3.32 For product weights, show that the weighted L_2-discrepancy is tractable, if and only if

$$\limsup_{s \to \infty} \frac{\sum_{j=1}^s \gamma_j}{\log s} < \infty.$$

4

Nets and sequences

In this chapter we give an introduction to the concept of (t, m, s)-nets and $(\mathbf{T}, s)$-sequences. Compared to classical types of point sets and sequences, like Hammersley point sets or van der Corput-Halton sequences, the general concept of (t, m, s)-nets and $(\mathbf{T}, s)$-sequences is a more natural one. Whereas in former examples a certain generation algorithm was the centre and origin of the investigation, here the starting point is the central property of uniform distribution modulo one that all intervals have to contain the correct portion of points of a sequence. With this definition in mind we search for a reasonably large class of intervals which are "fair" in this sense with respect to a finite point set. This leads to the definition of (t, m, s)-nets and their infinite analogues, to $(\mathbf{T}, s)$-sequences.

The generation of such point sets and sequences is mainly based on the digital construction scheme which leads to the notion of digital nets and sequences. Although such constructions go back to Sobol′ [251] and Faure [66] the detailed introduction and investigation of the general concept was given by Niederreiter [170]. This paper can be seen nowadays as the initiation of the whole theory of (t, m, s)-nets and $(\mathbf{T}, s)$-sequences. An introduction can also be found in [175, Chapter 4].

4.1 Motivation, fair intervals

The origin of studying (t, m, s)-nets, and, more generally, "fair intervals", is the property of uniform distribution modulo one (see Definition 3.1). For a finite point set $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$ in $[0, 1)^s$ it is never possible that it is absolutely uniformly distributed. That is, there are always subsets J, moreover there are always even intervals J in $[0, 1)^s$, for which

$$\frac{A(J, N, \mathcal{P})}{N} = \lambda_s(J)$$

does not hold. For instance, take an interval J of positive volume containing none of the points $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}$ (see Figure 4.1). Then $A(J, N, \mathcal{P})/N = 0 < \lambda_s(J)$. If $\mathcal{P}$ is finite such intervals J can always be found.

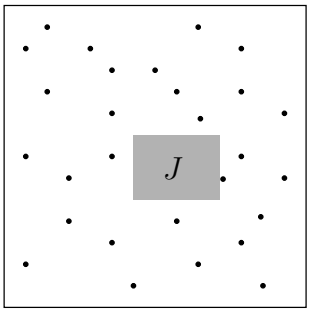

Figure 4.1 An interval J containing no point of $\mathcal{P}$.

Let us use the following notation.

Definition 4.1 For a given set $\mathcal{P}$ consisting of N points in $[0,1)^s$, we say for a subset J of $[0,1)^s$ that it is *fair* (with respect to $\mathcal{P}$), if

$$\frac{A(J, N, \mathcal{P})}{N} = \lambda_s(J).$$

This notation is also used the other way round.

Definition 4.2 For a given subset J of $[0,1)^s$, we say that a set $\mathcal{P}$ consisting of N points in $[0,1)^s$ is *fair* (with respect to J), if

$$\frac{A(J, N, \mathcal{P})}{N} = \lambda_s(J).$$

As we have seen, it is never possible that all intervals J are fair with respect to a given finite point set $\mathcal{P}$. Indeed from the result of Roth, see Theorem 3.20, it follows that there even always exists an interval J with

$$\left| \frac{A(J, N, \mathcal{P})}{N} - \lambda_s(J) \right| \geq c_s \frac{(\log N)^{(s-1)/2}}{N}$$

with a constant $c_s > 0$, depending only on the dimension s. However, for given s and N we could try to consider a certain class $\mathcal{C}$ of intervals J in $[0,1)^s$ and to find point sets $\mathcal{P}$ in $[0,1)^s$ such that any $J \in \mathcal{C}$ is fair with respect to $\mathcal{P}$.

Definition 4.3 For a given set $\mathcal{C}$ of subsets J of $[0,1)^s$, $\mathcal{C} \subseteq \{J : J \subseteq [0,1)^s\}$, we say that a set $\mathcal{P}$ consisting of N points in $[0,1)^s$ is *fair* (with respect to $\mathcal{C}$), if

$$\frac{A(J,N,\mathcal{P})}{N} = \lambda_s(J) \quad \text{for all } J \in \mathcal{C}.$$

Of course we would like to consider classes $\mathcal{C}$ of intervals as large as possible, with the hope that then for all intervals J in $[0,1)^s$ the fraction $A(J,N,\mathcal{P})/N$ is at least approximately equal to $\lambda_s(J)$.

Let us consider one concrete example. Choose $s = 2, N = 16$ and

$$\mathcal{C} = \{J_{A,B} = [A/4,(A+1)/4) \times [B/4,(B+1)/4) \,:\, A,B \in \{0,1,2,3\}\},$$

see Figure 4.2. We remark that the choice of half-open intervals here and in the following is of minor importance.

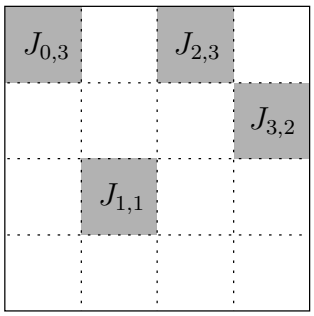

Figure 4.2 Intervals $J_{0,1}, J_{1,1}, J_{2,3}, J_{3,2}$ from the class $\mathcal{C}$ for $s = 2$ and $N = 16$.

If we choose for $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{15}\}$ the regular lattice (for convenience with the points centred in the intervals, see Figure 4.3), then clearly every $J_{A,B} \in \mathcal{C}$ is fair with respect to $\mathcal{P}$, i.e.,

$$\frac{A(J_{A,B},16,\mathcal{P})}{16} = \frac{1}{16} = \lambda_2(J_{A,B}).$$

Trivially, any interval J, which is a union of some of the disjoint intervals $J_{A,B}$ is fair as well. Consider, for example, $J := J_{0,1} \cup J_{1,1} \cup J_{2,1}$, for which we have $A(J,16,\mathcal{P})/16 = 3/16 = \lambda_2(J)$.

Instead of $\mathcal{C}$ we could even choose the larger class

$$\mathcal{C}_1 = \{[A/4,C/4) \times [B/4,D/4) \,:\, 0 \le A < C \le 4, 0 \le B < D \le 4\}$$

containing all intervals, which are unions of intervals of $\mathcal{C}$. Every interval

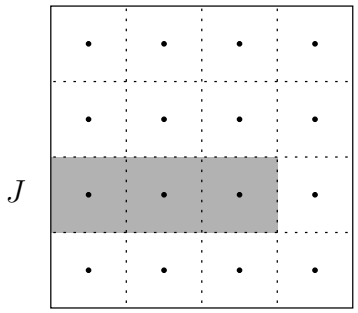

Figure 4.3 The class $\mathcal{C}$ for $s = 2$ and $N = 16$ and the regular lattice with 16 points.

in $\mathcal{C}_1$ is fair with respect to the regular lattice $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{15}\}$. But this fact does not give more information than the fact that the set $\mathcal{C}$ of "more elementary" intervals is fair. This consideration can be extended to arbitrary dimension s and to arbitrary N of the form $N = b^s$ with an integer $b \geq 2$. The regular lattice $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{b^s-1}\}$ then has the property that

$$\mathcal{C} := \left\{ \prod_{i=1}^{s} \left[\frac{A_i}{b}, \frac{A_i + 1}{b} \right) \,:\, 0 \leq A_i < b \text{ for } 1 \leq i \leq s \right\}$$

and therefore

$$\mathcal{C}_1 := \left\{ \prod_{i=1}^{s} \left[\frac{A_i}{b}, \frac{B_i}{b} \right) \,:\, 0 \leq A_i < B_i \leq b \text{ for } 1 \leq i \leq s \right\}$$

is fair with respect to $\mathcal{P}$. Thus we have a reasonably large class of fair intervals for the regular lattice.

However, we have already seen that the (star) discrepancy of the regular lattice (centred or not) is rather large (see Proposition 3.32 and Remark 3.33). Consider, for example, the rather large intervals $J_1 = [0, \frac{1}{8}) \times [0, 1)$ or $J_2 = [0, 1) \times (\frac{3}{8}, \frac{5}{8})$, which do not contain any point and so they are, by far, not fair (we should have $A(J_1, 16, \mathcal{P}) = 2$ and $A(J_2, 16, \mathcal{P}) = 4$). In general, the interval

$$J = \prod_{i=1}^{s-1} [0, 1) \times \left(\frac{1}{2}\frac{1}{b}, \frac{3}{2}\frac{1}{b} \right)$$

is empty, whereas we should have $A(J, b^s, \mathcal{P}) = b^{s-1}$, and hence we have

$$D_N(\mathcal{P}) \geq \left| \frac{A(J, N, \mathcal{P})}{N} - \lambda_s(J) \right| = \left| 0 - \frac{1}{b} \right| = \frac{1}{b} = \frac{1}{N^{1/s}}.$$

This means that to obtain point sets with a small (star) discrepancy we certainly have to demand fairness for larger, and in some sense, finer classes $\mathcal{C}$ of intervals.

Now let us try to extend $\mathcal{C}$ to a class $\widetilde{\mathcal{C}}$ such that fairness still can be attained with respect to certain point sets.

For simplicity let us again restrict ourselves to half-open intervals. Since for $J \in \widetilde{\mathcal{C}}$ we demand $A(J, 16, \mathcal{P})/16 = \lambda_s(J)$, we must have $\lambda_s(J) = k/16$ for an integer $k \geq 1$. Since any interval J of volume $k/16$ can be represented by the union of disjoint intervals of volume $1/16$, let us restrict ourselves to intervals of volume $1/16$.

Examples of such intervals are intervals of the "elementary" form $[0,1) \times [\frac{B}{16}, \frac{B+1}{16})$ or $[\frac{A}{2}, \frac{A+1}{2}) \times [\frac{B}{8}, \frac{B+1}{8})$ or $[\frac{A}{4}, \frac{A+1}{4}) \times [\frac{B}{4}, \frac{B+1}{4})$ and similar ones (see Figure 4.4).

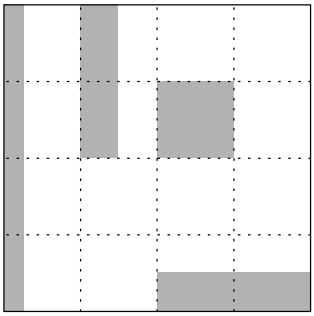

Figure 4.4 "Elementary" intervals of area $1/16$.

Considering these intervals means a considerable extension of the class $\mathcal{C}$. Obviously there are many other intervals of volume $1/16$, for example $J_1 := [0, \frac{1}{\pi}) \times [0, \frac{\pi}{16})$ or $J_2 := [\frac{7}{16}, \frac{7}{16} + \frac{1}{4}) \times [\frac{7}{16}, \frac{7}{16} + \frac{1}{4})$, see Figure 4.5.

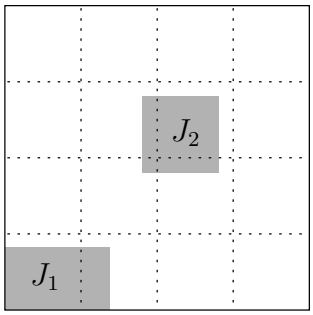

Figure 4.5 The intervals J_1 and J_2.

It is quite obvious that including intervals of the "J_1-type" (any interval of prescribed volume) together with the "elementary" intervals in $\widetilde{\mathcal{C}}$ would cause problems with finding a point set $\mathcal{P}$ in $[0,1)^s$, which is fair for all these $J \in \widetilde{\mathcal{C}}$. Although it is not so obvious, also including intervals of the form J_2 (all translates of intervals from $\widetilde{\mathcal{C}}$) together with the elementary intervals in $\widetilde{\mathcal{C}}$, at least in most cases, is a too restrictive demand (see the example below).

Let us consider

$$\widetilde{\mathcal{C}} := \left\{ \left[\frac{A}{2^d}, \frac{A+1}{2^d}\right) \times \left[\frac{B}{2^{4-d}}, \frac{B+1}{2^{4-d}}\right) : \right.$$
$$\left. d \in \{0,1,2,3,4\},\ 0 \le A < 2^d,\ 0 \le B < 2^{4-d} \right\}.$$

Obviously $\mathcal{C} \subseteq \widetilde{\mathcal{C}}$. The question is the following. Is there a point set $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{15}\}$ in $[0,1)^2$ which is fair with respect to $\widetilde{\mathcal{C}}$? That is, such that any $J \in \widetilde{\mathcal{C}}$ contains exactly one point of $\mathcal{P}$? The answer is *yes*! Take, for example, the 2-dimensional 16-point Hammersley point set in base 2 from Definition 3.44, see Figure 4.6.

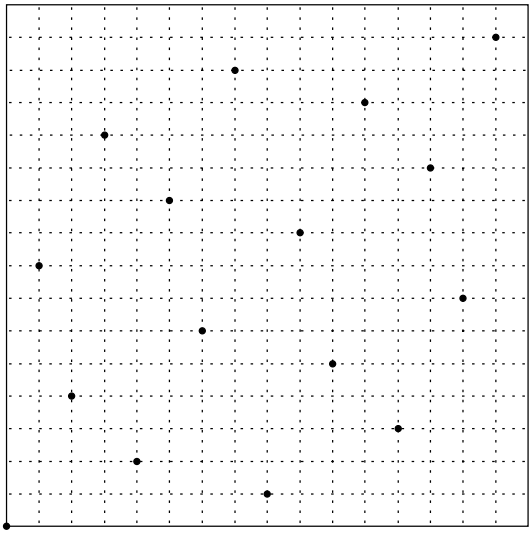

Figure 4.6 The 16 point Hammersley point set in base 2.

We shall show now that it is not possible to satisfy the fairness condition if we include also intervals of the type J_2 in $\widetilde{\mathcal{C}}$.

Consider the left lower quarter $[0,1/2)^2$ of the unit square. It must contain exactly four points. Any of the four (right half-open) rows R_1, R_2, R_3, R_4

and any of the four (right half-open) columns C_1, C_2, C_3, C_4, indicated in Figure 4.7, must contain exactly one point.

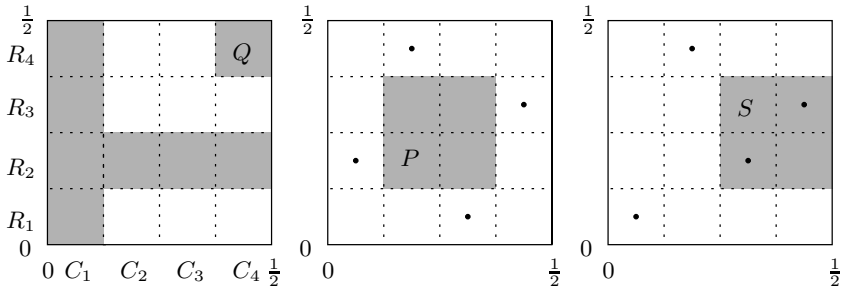

Figure 4.7 Placing four points in $[0, 1/2)^2$ which are fair with respect to R_1, R_2, R_3, R_4 and to C_1, C_2, C_3, C_4.

In what way ever one tries to place four points in the above square (see again Figure 4.7), there are either two small sub-squares of type Q, containing one point each, which are joined at one vertex, or the square P remains empty. The intervals S and P are of type J_2, are of volume $1/16$, and therefore should contain exactly one point. Hence it is useless to demand the fairness condition for a class of intervals containing the elementary as well as also the J_2-type intervals.

In general it is reasonable to ask the following. Given a dimension s and a number N, which is a power of any integer base $b \geq 2$, say $N = b^m$ (in the above example we restricted $N = b^s$), is there always a point set $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$ in $[0, 1)^s$ which is fair with respect to the class $\widetilde{\mathcal{C}}$ of elementary intervals of order m (see Definition 3.8)? That is, is there a point set $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$ which is fair with respect to

$$\widetilde{\mathcal{C}} = \left\{ \prod_{i=1}^{s} \left[\frac{A_i}{b^{d_i}}, \frac{A_i + 1}{b^{d_i}} \right) : d_1, \ldots, d_s \in \mathbb{N}_0,\ d_1 + \cdots + d_s = m, \right.$$
$$\left. 0 \leq A_i < b^{d_i} \text{ for } 1 \leq i \leq s \right\}?$$

The answer is, in general, *no*. A proof of this fact was given by Sobol′ [251].

Example 4.4 We show that even for $s = 4$ and $N = 2^2$ ($b = 2$ and $m = 2$) such a point set does not exist. In our argument we follow the proof of this result in [251, Section 5.5].

Assume to the contrary that there are four points $\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$ which are fair with respect to the corresponding $\widetilde{\mathcal{C}}$. For abbreviation let us write $(k_1 l_1, k_2 l_2, k_3 l_3, k_4 l_4)$ to denote the interval $\prod_{i=1}^{4} [k_i, l_i)$. The four-dimensional

unit cube $[0,1)^4$ is the union of the sixteen disjoint intervals

$$\begin{array}{ll}
(0\frac{1}{2},0\frac{1}{2},0\frac{1}{2},0\frac{1}{2}) & \text{type 0}\\
(\frac{1}{2}1,0\frac{1}{2},0\frac{1}{2},0\frac{1}{2}),\dots,(0\frac{1}{2},0\frac{1}{2},0\frac{1}{2},\frac{1}{2}1) & \text{type 1}\\
(\frac{1}{2}1,\frac{1}{2}1,0\frac{1}{2},0\frac{1}{2}),\dots,(0\frac{1}{2},0\frac{1}{2},\frac{1}{2}1,\frac{1}{2}1) & \text{type 2}\\
(\frac{1}{2}1,\frac{1}{2}1,\frac{1}{2}1,0\frac{1}{2}),\dots,(0\frac{1}{2},\frac{1}{2}1,\frac{1}{2}1,\frac{1}{2}1) & \text{type 3}\\
(\frac{1}{2}1,\frac{1}{2}1,\frac{1}{2}1,\frac{1}{2}1) & \text{type 4}
\end{array}$$

Because of symmetry we can assume without loss of generality that $\boldsymbol{x}_0 \in (0\frac{1}{2},0\frac{1}{2},0\frac{1}{2},0\frac{1}{2})$.

1. Assume that $\boldsymbol{x}_1$ is also an element of the type 0 interval or $\boldsymbol{x}_1$ is an element of a type 1 interval, without restriction of generality say $\boldsymbol{x}_1 \in (\frac{1}{2}1,0\frac{1}{2},0\frac{1}{2},0\frac{1}{2})$ or of a type 2 interval, without restriction of generality say $\boldsymbol{x}_1 \in (\frac{1}{2}1,\frac{1}{2}1,0\frac{1}{2},0\frac{1}{2})$. Then there are at least two points in the elementary interval $(01,01,0\frac{1}{2},0\frac{1}{2})$ of volume $\frac{1}{4}$ which must contain exactly one point.
2. Assume that $\boldsymbol{x}_1$ is an element of the type 4 interval $(\frac{1}{2}1,\frac{1}{2}1,\frac{1}{2}1,\frac{1}{2}1)$. Then with the same argument as above none of the points $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$ can be contained in a type 4 interval or a type 3 interval, and so there is no space at all for $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$.
3. Therefore $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ (and also $\boldsymbol{x}_3$) must be contained in type 3 intervals, without loss of generality assume they are contained in $(0\frac{1}{2},\frac{1}{2}1,\frac{1}{2}1,\frac{1}{2}1) \cup (\frac{1}{2}1,0\frac{1}{2},\frac{1}{2}1,\frac{1}{2}1)$. Then there are at least two points in the elementary interval $(01,01,\frac{1}{2}1,\frac{1}{2}1)$ of volume $\frac{1}{4}$ which must contain exactly one point.

Hence a fair distribution of four points in $[0,1)^4$, in the above sense, is not possible.

The answer to the question, when a fair distribution can be attained depends on the parameters b and s (and not on m) as is shown in the next section. Alas, in general, the demand for fairness for all intervals in $\widetilde{\mathcal{C}}$ must be weakened. A quite reasonable way to do this is the following. If $\mathcal{P} = \{\boldsymbol{x}_0,\dots,\boldsymbol{x}_{N-1}\}$ is fair with respect to all elementary intervals

$$\prod_{i=1}^{s}\left[\frac{A_i}{b^{d_i}},\frac{A_i+1}{b^{d_i}}\right)$$

of volume b^{-m} in $\widetilde{\mathcal{C}}$, i.e., $d_1+\cdots+d_s=m$, then of course it is also fair with respect to all intervals $\prod_{i=1}^{s}[\frac{A_i}{b^{d_i}},\frac{A_i+1}{b^{d_i}})$ with $d_1+\cdots+d_s \le m$, since any such interval is disjoint union of elementary intervals of volume b^{-m}.

For example,

$$\left[\frac{1}{2},1\right)\times\left[\frac{1}{2},1\right)=\bigcup_{k=1}^{4}\left(\left[\frac{1}{2},1\right)\times\left[\frac{3+k}{8},\frac{4+k}{8}\right)\right),$$

see Figure 4.8.

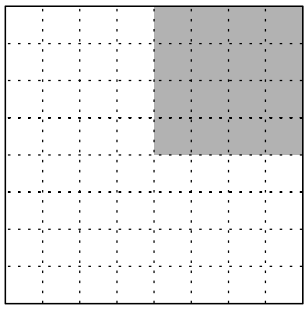

Figure 4.8 A disjoint union of elementary intervals.

To weaken the original condition

"fairness with respect to all $J=\prod_{i=1}^{s}[\frac{A_i}{b^{d_i}},\frac{A_i+1}{b^{d_i}})$ with $d_1+\cdots+d_s=m$",

i.e., to all b-adic elementary intervals of order m (see Definition 3.8), we could instead demand

"fairness with respect to all $J=\prod_{i=1}^{s}[\frac{A_i}{b^{d_i}},\frac{A_i+1}{b^{d_i}})$ with $d_1+\cdots+d_s=m-1$",

i.e., to all b-adic elementary intervals of order $m-1$. Obviously the first condition does contain the second condition, whereas the second condition does not contain the first one. To illustrate this, consider the example in Figure 4.9 for $s=2, b=2$ and $N=2^2$. The four points are fair to all 2-

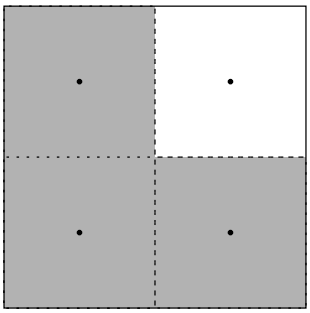

Figure 4.9 An example for $s=2, b=2$ and $N=2^2$.

adic elementary intervals of order 1 (area 1/2), but there is no point in the elementary interval $[0, \frac{1}{4}) \times [0, 1)$ of order 2 (area $1/2^2$). If this condition still cannot be satisfied, then we can again replace the order $m-1$ by $m-t$ for some $2 \le t \le m$. Finally, by choosing $t = m$, we obtain the condition that $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}$, with $N = b^m$, is fair with respect to $[0, 1)^s$, which is trivially satisfied.

We motivated these considerations by starting with the (centred) regular lattice. Let us finish this section with an example by considering the regular lattice (centred or not) once more with respect to the above condition on elementary intervals.

Let a dimension s and a base b be given and let $N = b^m$ be such that we can generate a (centred) regular lattice with N points. This is certainly possible if $m = Ls$ for a positive integer L. Then for points of the centred regular lattice we can choose the centres of the sub-cubes $\prod_{i=1}^{s} [\frac{A_i}{b^L}, \frac{A_i+1}{b^L}), 0 \le A_i < b^L$ for $1 \le i \le s$.

Example 4.5 For $s = b = L = 2$ we get the point set from Figure 4.10. This point set is not fair with respect to all 2-adic intervals of order 4 or

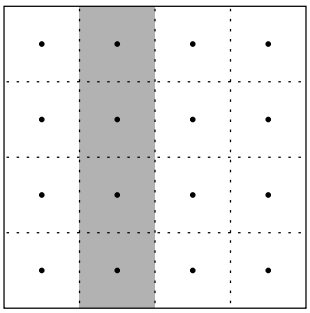

Figure 4.10 Centred regular lattice Γ_4^{c} with 16 points and an elementary interval of order 2.

of order 3. For example, the elementary interval $[0, \frac{1}{8}) \times [0, 1)$ of order 3 (area $1/2^3$) contains no point of the (centred) regular lattice. However, it is fair with respect to all elementary intervals of order 2 (area $1/2^2$) and of lower order, since any elementary interval of order 2 is a (disjoint) union of sub-cubes $[\frac{A_1}{4}, \frac{A_1+1}{4}) \times [\frac{A_2}{4}, \frac{A_2+1}{4})$ with $0 \le A_1, A_2 < 4$, all of which contain one point, have area $1/2^4$ and are therefore fair (see Figure 4.10).

In general we have the following result.

Lemma 4.6 *The (centred) regular lattice of b^{Ls} points in $[0, 1)^s$ is fair for*

the class of all b-adic elementary intervals of order L. It is not fair for the class of all b-adic elementary intervals of order L + 1.

Proof The b-adic elementary interval $[0, \frac{1}{b^{L+1}}) \times \prod_{i=2}^{s}[0,1)$ of order $L+1$ is not fair with respect to the regular lattice. Any b-adic elementary interval J of order L, say $J = \prod_{i=1}^{s}[\frac{A_i}{b^{d_i}}, \frac{A_i+1}{b^{d_i}})$ with $d_1 + \cdots + d_s = L$ and $0 \le A_i < b^{d_i}$ for $1 \le i \le s$, can be represented as the disjoint union of fair sub-cubes by

$$J = \bigcup_{B_1=b^{L-d_1}A_1}^{b^{L-d_1}(A_1+1)-1} \cdots \bigcup_{B_s=b^{L-d_s}A_s}^{b^{L-d_s}(A_s+1)-1} \prod_{i=1}^{s}\left[\frac{B_i}{b^L}, \frac{B_i+1}{b^L}\right).$$

Therefore J is fair and the result follows. □

In this section we have provided the motivation for the definition of a (t, m, s)-net in base b, which is given in the next section.

4.2 (t, m, s)-nets and their basic properties

Motivated by the discussion in the previous section we give the following definitions (thereby we essentially follow the general definitions given for the first time by Niederreiter in [170]).

Recall that, according to Definition 3.8, for a given dimension $s \ge 1$, an integer base $b \ge 2$, and a nonnegative integer k, a b-adic s-dimensional elementary interval of order k is an interval of the form

$$J = \prod_{i=1}^{s}\left[\frac{A_i}{b^{d_i}}, \frac{A_i+1}{b^{d_i}}\right),$$

where $d_1, \ldots, d_s \in \mathbb{N}_0$ with $d_1 + \cdots + d_s = k$ and $0 \le A_i < b^{d_i}$ for $1 \le i \le s$.

Definition 4.7 For a given dimension $s \ge 1$, an integer base $b \ge 2$, a positive integer m, and an integer t with $0 \le t \le m$, a point set $\mathcal{P}$ of b^m points in $[0,1)^s$ is called a (t, m, s)-*net in base* b if the point set $\mathcal{P}$ is fair with respect to all b-adic s-dimensional elementary intervals of order $m - t$.

Definition 4.8 A (t, m, s)-net in base b with $t \ge 1$ is called a *strict* (t, m, s)-net in base b if it is not a $(t-1, m, s)$-net in base b. Furthermore a $(0, m, s)$-net in base b is called strict by definition.

Remark 4.9 1. The property for $\mathcal{P}$ to be a (t, m, s)-net in base b means that every interval $J = \prod_{i=1}^{s}[\frac{A_i}{b^{d_i}}, \frac{A_i+1}{b^{d_i}})$ with $d_1 + \cdots + d_s = m - t$, that is, of volume b^{-m+t}, contains exactly b^t points of $\mathcal{P}$.

2. Since for every $k \geq 1$ every b-adic s-dimensional elementary interval of order $k-1$ (volume b^{-k+1}) is the union of b disjoint b-adic s-dimensional elementary intervals of order k, every (t, m, s)-net in base b with $t \leq m-1$ is also a $(t+1, m, s)$-net in base b.
3. Every point set of b^m points in $[0,1)^s$ is an (m, m, s)-net in base b. The condition then is that the interval $J = [0,1)^s$ contains b^m points of the set, which is trivially satisfied.
4. It does not make sense to define the notion of (t, m, s)-nets in base b for negative t, since a point set of b^m points can never be fair with respect to an interval of volume less than b^{-m}.
5. We call t the *quality parameter* of the (t, m, s)-net.

First examples

We provide two examples for (t, m, s)-nets.

Example 4.10 As a first nontrivial example let us consider a (centred) regular lattice $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$ of $N = b^{sL}$ points in $[0,1)^s$. Letting $m = sL$, the point set is in any case an (m, m, s)-net in base b. But, by Lemma 4.6, we have that $\mathcal{P}$ is fair with respect to every b-adic s-dimensional elementary interval of order L, and this order L is optimal. Consequently we get the following corollary from Lemma 4.6.

Corollary 4.11 *The (centred) regular lattice of b^m points, with $m = sL$, in $[0,1)^s$ is a strict $(m(1-\frac{1}{s}), m, s)$-net in base b.*

Remark 4.12 Intuitively, the strict quality parameter $t = m(1-1/s)$ in the scale between 0 and m is rather large for dimension $s \geq 3$. This fits with the bad order of magnitude of the (star) discrepancy of the regular lattice in dimensions larger than or equal to three. For $s = 1$ we obtain an equidistant point set in $[0,1)$ of optimal star discrepancy $1/(2N)$, which fits with the optimal quality parameter $t = 0$. For $s = 2$, the regular lattice has a discrepancy of order $1/\sqrt{N}$, an order which essentially coincides with the average order of the discrepancy of N-element point sets in $[0,1)^2$. This again fits with the median value $m/2$ for the quality parameter t. (We remark that these results also hold for the 'noncentred' regular lattice.)

As a second example let us consider a two-dimensional Hammersley point set in base b, see Definition 3.44.

Lemma 4.13 *For a given base b and a given positive integer m, the two-dimensional Hammersley point set $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$ with $N = b^m$ and $\boldsymbol{x}_k = (k/N, \varphi_b(k))$ for $0 \leq k \leq N-1$ is a $(0, m, 2)$-net in base b.*

Proof First recall the definition of the b-adic radical inverse function φ_b. For a nonnegative integer k with b-adic expansion $k = \kappa_{r-1}b^{r-1} + \kappa_{r-2}b^{r-2} + \cdots + \kappa_1 b + \kappa_0$ we define

$$\varphi_b(k) := \frac{\kappa_0}{b} + \frac{\kappa_1}{b^2} + \cdots + \frac{\kappa_{r-1}}{b^r} \in [0,1).$$

For a b-adic elementary two dimensional interval J of order m, i.e.,

$$J = \left[\frac{A_1}{b^d}, \frac{A_1+1}{b^d}\right) \times \left[\frac{A_2}{b^{m-d}}, \frac{A_2+1}{b^{m-d}}\right)$$

with $0 \le A_1 < b^d$ and $0 \le A_2 < b^{m-d}$, we have to determine the number of $\boldsymbol{x}_k$ contained in J. Recall that this number should be one.

Note that for k with $0 \le k < b^m$ and b-adic representation $k = \kappa_{m-1}b^{m-1} + \cdots + \kappa_0$ the point $\boldsymbol{x}_k$ belongs to J if and only if

$$\frac{k}{b^m} \in \left[\frac{A_1}{b^d}, \frac{A_1+1}{b^d}\right) \quad \text{and} \quad \varphi_b(k) \in \left[\frac{A_2}{b^{m-d}}, \frac{A_2+1}{b^{m-d}}\right).$$

This is the case if and only if

$$A_1 b^{m-d} \le \kappa_{m-1}b^{m-1} + \cdots + \kappa_0 < A_1 b^{m-d} + b^{m-d}$$

and

$$A_2 b^d \le \kappa_0 b^{m-1} + \cdots + \kappa_{m-1} < A_2 b^d + b^d.$$

By the first condition the digits $\kappa_{m-d}, \ldots, \kappa_{m-1}$ are uniquely determined (whereas the digits $\kappa_{m-d-1}, \ldots, \kappa_0$ can be chosen arbitrarily). By the second condition the digits $\kappa_0, \ldots, \kappa_{m-d-1}$ are uniquely determined (whereas the digits $\kappa_{m-d}, \ldots, \kappa_{m-1}$ can be chosen arbitrarily). Hence there is a uniquely determined k such that $\boldsymbol{x}_k \in J$. □

A $(0, m, s)$-net in base b does not exist for all parameters m,s, and b. For instance, in Example 4.4 it was shown that there does not exist a $(0, 2, 4)$-net in base 2. Consequently we show below that there does not exist a $(0, m, s)$-net in base 2 for any $m \ge 2$ and $s \ge 4$. Before we do so, we convince ourselves of several, so-called, *propagation rules* for (t, m, s)-nets. Here, a propagation rule is a method of constructing new (t, m, s)-nets from other, given, (t, m, s)-nets.

Propagation rules for nets

Note that a (t, m, s)-net in base b easily looses its quality by elementary movements. It does not loose its net property entirely since any point set of

b^m points in $[0,1)^s$ is a (m, m, s)-net in base b. However, its quality parameter t has no stability with respect to even simple movements. For instance, consider the $(0, 2, 2)$-net in base 2 from Figure 4.11 (left picture) and apply a translation along b or a reflection on a, considered modulo one in each coordinate (see Figure 4.11). Then both new point sets are now strict $(2, 2, 2)$-nets in base 2. As we see below, more stability can be found for so-called digital nets (see Lemma 4.63 in Section 4.4).

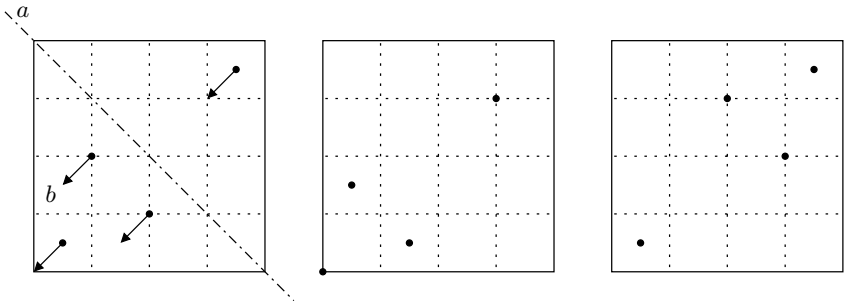

Figure 4.11 $(0, 2, 2)$-net in base 2 with elementary movements.

We may ask what happens with the net-structure if we merge (t, m, s)-nets in base b to one point set. Assume we have r point sets $\mathcal{P}_1, \ldots, \mathcal{P}_r$, where $\mathcal{P}_i$ is a (t_i, m_i, s)-net in base b. Assume further that $b^{m_1} + \cdots + b^{m_r} = b^m$ for some integer m. Then the multiset union $\mathcal{P} := \mathcal{P}_1 \cup \ldots \cup \mathcal{P}_r$ is of course a (t, m, s)-net in base b, at least for $t = m$. But we can say even more.

Lemma 4.14 *For $1 \le j \le r$ let $\mathcal{P}_j$ be (t_j, m_j, s)-nets in base b, with $m_1, \ldots, m_r$ such that $b^{m_1} + \cdots + b^{m_r} = b^m$ for some integer m. Then the multiset union $\mathcal{P} := \mathcal{P}_1 \cup \ldots \cup \mathcal{P}_r$ is a (t, m, s)-net in base b with*

$$t = m - \min_{1 \le j \le r} (m_j - t_j).$$

Proof Let J be an elementary interval in base b of order $w := \min_{1 \le j \le r}(m_j - t_j)$. For every $1 \le j \le r$, J contains exactly $b^{m_j - w}$ of the elements of $\mathcal{P}_j$. Note that any interval of order less than or equal to $m_j - t_j$ is fair with respect to $\mathcal{P}_j$ and that $w \le m_j - t_j$. Hence J contains exactly $\sum_{j=1}^{r} b^{m_j - w} = b^{m-w}$ elements of $\mathcal{P}$ and is therefore fair with respect to $\mathcal{P}$. Consequently, the strict quality parameter t of $\mathcal{P}$ is at most $m - w$ and the result follows. □

Remark 4.15 For example, the superposition of b^r copies of a (t, m, s)-net in base b yields a $(t + r, m + r, s)$-net in base b. This is [190, Lemma 10].

Let $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{b^m - 1}\}$ be a (t, m, s)-net in base b and let $1 \le n \le s$.

Fix now any n of the s dimensions, without restriction of generality, say, the first n dimensions. For every $\boldsymbol{x}_k$ we now consider only the first n coordinates. We obtain an n-dimensional point set, say, $\widetilde{\mathcal{P}} = \{\boldsymbol{y}_0, \ldots, \boldsymbol{y}_{b^m-1}\}$. We then have the following lemma.

Lemma 4.16 *Let $\mathcal{P}$ be a (t, m, s)-net in base b and let $\widetilde{\mathcal{P}}$ be defined as above. Then the point set $\widetilde{\mathcal{P}}$ is a (t, m, n)-net in base b.*

Proof Let $\widetilde{J}$ be a b-adic n-dimensional elementary interval of order $m - t$, then $J := \widetilde{J} \times [0, 1)^{s-n}$ is a b-adic s-dimensional elementary interval of order $m-t$ and hence J contains exactly b^t of the $\boldsymbol{x}_h$, $0 \leq h < b^m$. Since J puts no conditions on the last $s - n$ coordinates this means that $\widetilde{J}$ contains exactly b^t points of the point set $\widetilde{\mathcal{P}}$ (see Figure 4.12 for an example). □

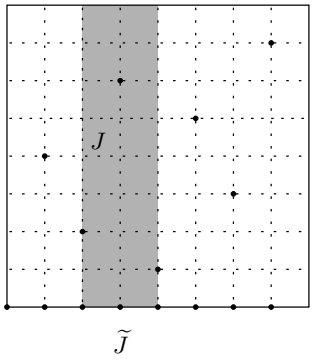

Figure 4.12 Projection of a $(0, 3, 2)$-net in base 2 to the first components.

The above result cannot be improved in the following sense. If $\mathcal{P}$ is a strict (t, m, s)-net in base b, then we cannot conclude that $\widetilde{\mathcal{P}}$ is also a strict (t, m, n)-net in base b. An extreme example is the following.

Let $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{b^m-1}\}$ be defined by $\boldsymbol{x}_k = (0, \frac{k}{b^m})$ for $0 \leq k < b^m$. Then $\mathcal{P}$ is a strict $(m, m, 2)$-net in base b. Its first projection is a strict $(m, m, 1)$-net in base b and its second projection is a $(0, m, 1)$-net in base b (see Figure 4.13).

We have now propagation rules concerning t and s (see also the collection of the propagation rules in Chapter 9). In the following we provide a propagation rule concerning m.

It is not true in general that for a (t, m, s)-net $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{b^m-1}\}$ in base b the truncated point set $\widetilde{\mathcal{P}} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{b^r-1}\}$, for some r with $t < r < m$, is a (t, r, s)-net in base b. (Note that the case $r \leq t$ is trivial since every point set of b^r points is a (r, r, s)-net in base b.)

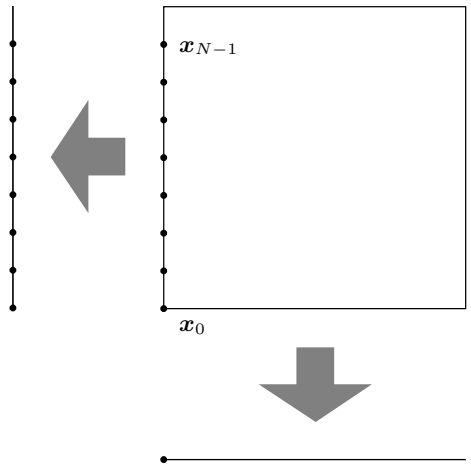

Figure 4.13 Projection of a $(m, m, 2)$-net to the first and second component.

The question arises now how to propagate in this case. We use the following approach (for an illustration see Figure 4.14):

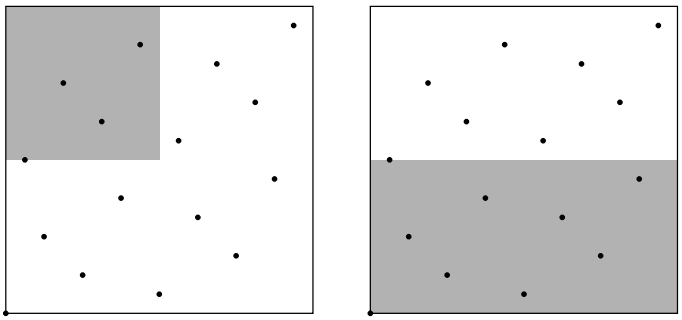

Figure 4.14 A $(0, 4, 2)$-net in base 2. The points in the elementary interval from the left picture yield a $(0, 2, 2)$-net in base 2 after doubling both coordinates. The points in the elementary interval from the right picture yield a $(0, 3, 2)$-net in base 2 after doubling the second coordinate.

1. Let $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{b^m-1}\}$ be a (t, m, s)-net in base b, and let $t < r < m$.
2. Take any elementary interval $J = \prod_{i=1}^{s} [\frac{A_i}{b^{d_i}}, \frac{A_i+1}{b^{d_i}})$ of order $m - r$, i.e., with $d_1 + \cdots + d_s = m - r$. Since $m - r < m - t$ this interval contains exactly b^r elements of the net $\mathcal{P}$.
3. We now translate the point $(\frac{A_1}{b^{d_1}}, \ldots, \frac{A_s}{b^{d_s}})$ of J to the origin and blow up the translated J and the translated net-points in J to the unit cube. That

is, we apply the affine transformation $T : \mathbb{R}^s \to \mathbb{R}^s$,

$$\boldsymbol{x} := (x_1, \ldots, x_s) \mapsto \left(b^{d_1} \left(x_1 - \frac{A_1}{b^{d_1}} \right), \ldots, b^{d_s} \left(x_s - \frac{A_s}{b^{d_s}} \right) \right)$$

to J and to the net points in J.

4. The point set obtained in this way, consisting of b^r points, is denoted by $\widetilde{\mathcal{P}} = \{\boldsymbol{y}_0, \ldots, \boldsymbol{y}_{b^r-1}\}$.

We claim that $\widetilde{\mathcal{P}}$ forms a (t, r, s)-net in base b (see also [175, Lemma 4.4]).

Lemma 4.17 *Let $\mathcal{P}$ be a (t, m, s)-net in base b, let $t < r < m$ and let J be an elementary interval of order $m - r$. Let T be an affine transformation of J onto $[0, 1)^s$. Then the points of $\mathcal{P}$ that belong to J are transformed by T into a (t, r, s)-net $\widetilde{\mathcal{P}}$ in base b.*

Proof Let J' be a b-adic s-dimensional elementary interval of order $r - t$. The number of points $\boldsymbol{y}_k$ from $\widetilde{\mathcal{P}}$ contained in J' equals the number of points $\boldsymbol{x}_k$ from $\mathcal{P}$ contained in the b-adic s-dimensional elementary interval $T^{-1}(M')$ of order $(r - t) + (m - r) = m - t$. This number is exactly b^t since $\mathcal{P}$ is a (t, m, s)-net in base b. □

Existence of $(0, m, s)$*-nets in base* b

As a corollary from Lemma 4.16 and Lemma 4.17 we obtain the following corollary.

Corollary 4.18 *A $(0, m, s)$-net in base 2 cannot exist if $m \geq 2$ and $s \geq 4$.*

Proof If a $(0, m, s)$-net in base 2 with $m \geq 2$ and $s \geq 4$ exists, then, by the above propagation rules (Lemmas 4.16 and 4.17) on m and s, a $(0, 2, 4)$-net in base 2 would exist, which is a contradiction in view of Example 4.4. □

Obviously, the point set $\left\{\boldsymbol{x}_0 = (0, \ldots, 0), \boldsymbol{x}_1 = (\frac{1}{2}, \ldots, \frac{1}{2})\right\}$ forms a $(0, 1, s)$-net in base 2 for all s. For $s = 2$, the two-dimensional Hammersley point set with 2^m points gives, for any integer $m \geq 1$, a $(0, m, 2)$-net in base 2 by Lemma 4.13. Hence, concerning the existence of $(0, m, s)$-nets in base 2, the only question remaining is whether there exist $(0, m, 3)$-nets in base 2 for all $m \geq 2$. This question was answered in the affirmative by Sobol′ [251]. Concrete examples of $(0, m, 3)$-nets in base 2 for any $m \geq 2$ are given in Section 4.4. (The examples given there are also special cases of nets obtained from Sobol′-, Faure- and Niederreiter sequences, see Chapter 8).

In arbitrary base $b \geq 2$ we have the following result, which for the first time in this form was shown by Niederreiter (see [175, Corollary 4.21]).

Corollary 4.19 *A $(0, m, s)$-net in base b cannot exist if $m \geq 2$ and $s \geq b + 2$.*

This corollary is a consequence of the following lemma:

Lemma 4.20 *A $(0, 2, b + 2)$-net in base $b \geq 2$ cannot exist.*

Proof Assume to the contrary that a $(0, 2, b + 2)$-net $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{b^2-1}\}$ in base b exists. Then any elementary interval of the form

$$[0,1)^i \times \left[\frac{A}{b}, \frac{A+1}{b}\right) \times [0,1)^j \times \left[\frac{B}{b}, \frac{B+1}{b}\right) \times [0,1)^{b-i-j}$$

of volume b^{-2} contains, by the net property, exactly one point of $\mathcal{P}$. We call this the "orthogonality property" of this net.

Let us check in which interval of the form $[A/b, (A+1)/b)$ of length b^{-1}, where $A \in \{0, \ldots, b-1\}$, each coordinate of each net point $\boldsymbol{x}_n$ contained. That is, we describe any $\boldsymbol{x}_n$ by a vector

$$\boldsymbol{x}_n \leftrightarrow \begin{pmatrix} a_n^{(1)} \\ \vdots \\ a_n^{(b+2)} \end{pmatrix},$$

where $a_n^{(i)} \in \{0, 1, \ldots, b-1\}$ is chosen such that the ith coordinate $x_{n,i}$ of $\boldsymbol{x}_n$ is contained in the interval $[a_n^{(i)}/b, (a_n^{(i)} + 1)/b)$.

Let us set these b^2 column vectors side by side, so we get an array of numbers of the form

$$\begin{array}{cccc} \boldsymbol{x}_0 & \boldsymbol{x}_1 & \ldots & \boldsymbol{x}_{b^2-1} \\ \updownarrow & \updownarrow & & \updownarrow \\ a_0^{(1)} & a_1^{(1)} & \ldots & a_{b^2-1}^{(1)} \\ \vdots & \vdots & & \vdots \\ a_0^{(b+2)} & a_1^{(b+2)} & \ldots & a_{b^2-1}^{(b+2)} \end{array} \tag{4.1}$$

Let us now take any two of the rows of the above array, say

$$\begin{array}{cccc} a_0^{(i)} & a_1^{(i)} & \ldots & a_{b^2-1}^{(i)}, \\ a_0^{(j)} & a_1^{(j)} & \ldots & a_{b^2-1}^{(j)}, \end{array}$$

then the "orthogonality property" of the net is equivalent to the fact that the above b^2 two-dimensional columns

$$\begin{pmatrix} a_k^{(i)} \\ a_k^{(j)} \end{pmatrix}_{k=0,\ldots,b^2-1}$$

attain any possible value $\binom{n}{r}$, with $n, r \in \{0, \ldots, b-1\}$, exactly once. Therefore, in particular, any possible value $n \in \{0, \ldots, b-1\}$ must occur in any row exactly b-times.

However, we show that this property cannot be satisfied for all possible pairs of rows.

Assume to the contrary that any two of the $b+2$-rows satisfy the orthogonality property. Without restriction of generality we can assume that in array (4.1) the values of the first column all equal 1 (a permutation of the values $r \in \{0, \ldots, b-1\}$ in a single row of the array does not affect the "orthogonal property").

Then in any of the remaining $b^2 - 1$ columns 1 can occur at most once. However, since in each row 1 must occur b-times, we would need place for $(b-1)(b+2)$ remaining 1's in these $b^2 - 1$ columns. Since $(b-1)(b+2) = b^2 + b - 2 > b^2 - 1$, we obtain a contradiction. □

Again, it is easy to provide a $(0, 1, s)$-net in base b for any dimension s. Faure- and Niederreiter sequences (see Chapter 8) provide, for any prime-power base b, any $m \geq 2$, and any $s \leq b+1$, examples of $(0, m, s)$-nets in base b. Hence the question concerning the existence of $(0, m, s)$-nets in base b is solved for all prime-power bases b. In general, it is not solved for composite bases b. It is known that the maximal dimension s for which there exists a $(0, m, s)$-net in base b with $m \geq 2$, for composite b, is much smaller than $b+1$. For more information see [139, Section 3] and the MinT database to be found at

http://mint.sbg.ac.at/

We just point out the following singular result.

Lemma 4.21 *There does not exist a* $(0, 2, 4)$*-net in base* 6.

The above considerations can be formulated and proved more elegantly in terms of combinatorial objects like orthogonal Latin squares or ordered orthogonal arrays. This is done in Chapter 6.

Further propagation rules for nets

We have already shown propagation rules for the parameters t,m, and s of a (t, m, s)-net in base b. In the following we consider possible propagation rules for the parameter b, the base of the net.

Such propagation rules principally should be of the following form.

1. Any (t, m, s)-net in base b is a (t', m', s')-net in base b', or
2. if there exist (t_j, m_j, s_j)-nets in bases b_j for $1 \le j \le l$, then there exists a (t', m', s')-net in base b'.

Note that for results of the first form, there must be a principal connection between the compared bases b and b', since the number of points b^m, respectively $b'^{m'}$ remains unchanged. That is $b^m = b'^{m'}$. Therefore b and b' must have the same prime divisors, say $b = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ and $b' = p_1^{\beta_1} \cdots p_r^{\beta_r}$ with $\alpha_i, \beta_i \ge 1$ for $1 \le i \le r$. Since $b^m = b'^{m'}$ we get $m\alpha_i = m'\beta_i$ for $1 \le i \le r$. Let d denote the greatest common divisor of m and m'. Then the integer $\widetilde{m} := m/d$ divides β_i and the integer $\widetilde{m}' := m'/d$ divides α_i, and $\gamma_i := \alpha_i/\widetilde{m}' = \beta_i/\widetilde{m}$ is a positive integer. Let $c = p_1^{\gamma_1} \cdots p_r^{\gamma_r}$, then $b = c^{\widetilde{m}'}$ and $b' = c^{\widetilde{m}}$. Therefore a simple propagation rule of the first kind can only exist if b and b' are powers of a common "base" c, say $b = c^L$ and $b' = c^{L'}$ with L and L' relative prime. Further, since then $c^{Lm} = b^m = b'^{m'} = c^{L'm'}$ that is $Lm = L'm'$, with $\gcd(L, L') = 1$ we must have $m = \mu L'$ and $m' = \mu L$ for some positive integer μ.

Consequently, propagation rules of the first kind have to be of the following principal form. Any $(t, \mu L', s)$-net in base c^L is a $(t', \mu L, s)$-net in base $c^{L'}$.

We do not give base-propagation rules of the second type here. But also the base-propagation rules of the first kind are of a more complex nature than the propagation rules on m,s, and t.

The simplest base propagation rule is based on the following fact concerning elementary intervals.

Lemma 4.22 *Let the integers $b, k \ge 2$ be given. Any b^k-adic s-dimensional elementary interval of order n is a b-adic s-dimensional elementary interval of order nk.*

Proof Let $J = \prod_{i=1}^{s} [\frac{A_i}{(b^k)^{d_i}}, \frac{A_i+1}{(b^k)^{d_i}})$ with $d_i \ge 0$ and $0 \le A_i < (b^k)^{d_i}$ for $1 \le i \le s$ and $d_1 + \cdots + d_s = n$ be an arbitrary b^k-adic s-dimensional elementary interval of order n. Define $d_i' := kd_i$, then $J = \prod_{i=1}^{s} [\frac{A_i}{b^{d_i'}}, \frac{A_i+1}{b^{d_i'}})$ with $d_i' \ge 0$, $0 \le A_i < b^{d_i'}$ for $1 \le i \le s$ and $d_1' + \cdots + d_s' = nk$ is an b-adic s-dimensional elementary interval of order nk. □

The converse, in general, does not hold. For instance the 2-adic 2-dimensional elementary intervals of order 2 are the intervals

$$
\begin{array}{llll}
[0,1) \times [0,\tfrac{1}{4}), & [0,1) \times [\tfrac{1}{4},\tfrac{1}{2}), & [0,1) \times [\tfrac{1}{2},\tfrac{3}{4}), & [0,1) \times [\tfrac{3}{4},1), \\
[0,\tfrac{1}{2}) \times [0,\tfrac{1}{2}), & [0,\tfrac{1}{2}) \times [\tfrac{1}{2},1), & [\tfrac{1}{2},1) \times [0,\tfrac{1}{2}), & [\tfrac{1}{2},1) \times [\tfrac{1}{2},1), \\
[0,\tfrac{1}{4}) \times [0,1), & [\tfrac{1}{4},\tfrac{1}{2}) \times [0,1), & [\tfrac{1}{2},\tfrac{3}{4}) \times [0,1), & [\tfrac{3}{4},1) \times [0,1),
\end{array}
$$

whereas the 4-adic 2-dimensional elementary intervals of order 1 are just the intervals in the first and in the third of the above lines.

Consequently we immediately obtain the following result.

Corollary 4.23 *Any $(t, \mu k, s)$-net in base b is a $(\lceil t/k \rceil, \mu, s)$-net in base b^k. The converse, in general, is not true.*

Proof Any b^k-adic s-dimensional elementary interval of order $\mu - \lceil t/k \rceil$ is a b-adic s-dimensional elementary interval of order $k(\mu - \lceil t/k \rceil) \leq k\mu - t$. Hence it is fair with respect to the $(t, \mu k, s)$-net in base b.

To provide a counterexample for the converse assertion, consider the $(0, 1, 2)$-net in base 4 given in Figure 4.15, which is not a $(0, 2, 2)$-net in base 2 (consider the corresponding elementary 2-adic and 4-adic intervals listed above) and the result follows. □

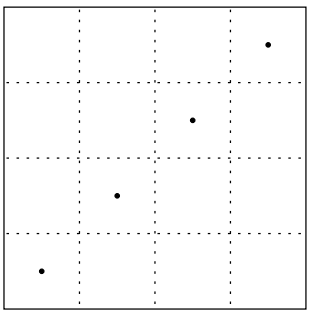

Figure 4.15 A $(0, 1, 2)$-net in base 4 which is not a $(0, 2, 2)$-net in base 2.

However, it is obvious that some relation must also hold in the converse direction and therefore, in general, between the quality parameters of a $(t, \mu L', s)$-net in base c^L and a $(t', \mu L, s)$-net in base $c^{L'}$.

The following base propagation rule was first given in [215], see also [216].

Theorem 4.24 *For given integers $c \geq 2$, L and $L' \geq 1$ with $\gcd(L, L') = 1$, for every dimension s, and all positive integers μ we have that every $(t, \mu L', s)$-net in base c^L is a $(t', \mu L, s)$-net in base $c^{L'}$, where*

$$t' = \min\left(\left\lceil \frac{Lt + \mu L(-L' \pmod L)}{L' + (-L' \pmod L)} \right\rceil, \left\lceil \frac{Lt + (s-1)(L-1)}{L'} \right\rceil\right).$$

Remark 4.25 Before we prove the theorem let us consider some special cases. For the trivial case of equal bases, i.e., for $L = L' = 1$, by the above estimate for t', we get the best possible result $t = t'$. For the case considered in Corollary 4.23, i.e., $L = 1$ by the above estimate for t' we get the best

possible result $t' = \lceil t/L' \rceil$. For $L' = 1$ we rewrite the above result and state it as a corollary on its own.

Corollary 4.26 *For a given base $b \geq 2$ and any integer $k \geq 1$, every (t, m, s)-net in base b^k is a (t', mk, s)-net in base b with*

$$t' = \min(t + m(k-1), kt + (s-1)(k-1)).$$

This result, for some cases, improves the corresponding result given by Niederreiter & Xing [190, Lemma 9], verifying a quality parameter $t' = \min(km, kt + (s-1)(k-1))$. Of course, $t + m(k-1) \leq km$ always. The result of the corollary is better than the result of Niederreiter & Xing if and only if $t + m(k-1) < kt + (s-1)(k-1)$ and if $t + m(k-1) < km$, i.e., if $m < s - 1 + t$ and if $t < m$.

Proof of Theorem 4.24 We write $m = \mu L'$ and $m' = \mu L$. Take an elementary interval $\widetilde{J}$ in base $c^{L'}$ of order, say $m' - t''$ (with some nonnegative integer t''), i.e., of volume $c^{-L'(m'-t'')}$, say

$$\widetilde{J} = \prod_{i=1}^{s} \left[\frac{A_i}{c^{L'd_i'}}, \frac{A_i + 1}{c^{L'd_i'}} \right)$$

with $d_1' + \cdots + d_s' = m' - t''$. For every $1 \leq i \leq s$ we set $L'd_i' = Ld_i - r_i$ with $0 \leq r_i < L$, then

$$\begin{aligned}
\widetilde{J} &= \prod_{i=1}^{s} \left[\frac{A_i c^{r_i}}{c^{Ld_i}}, \frac{A_i c^{r_i} + c^{r_i}}{c^{Ld_i}} \right) \\
&= \prod_{i=1}^{s} \bigcup_{k_i=0}^{c^{r_i}-1} \left[\frac{A_i c^{r_i} + k_i}{c^{Ld_i}}, \frac{A_i c^{r_i} + k_i + 1}{c^{Ld_i}} \right) \\
&= \bigcup_{k_1=0}^{c^{r_1}-1} \cdots \bigcup_{k_s=0}^{c^{r_s}-1} \prod_{i=1}^{s} \left[\frac{A_i c^{r_i} + k_i}{(c^L)^{d_i}}, \frac{A_i c^{r_i} + k_i + 1}{(c^L)^{d_i}} \right).
\end{aligned}$$

Therefore $\widetilde{J}$ is the union of elementary intervals in base c^L of order $d_1 + \cdots + d_s$ each. Therefore, $\widetilde{J}$ is fair with respect to a (t, m, s)-net $\mathcal{P}$ in base c^L if

$$d_1 + \cdots + d_s \leq m - t.$$

Substituting $L^{-1}(L'd_i' - r_i)$ for d_i, this is equivalent to

$$L' \sum_{i=1}^{s} d_i' + \sum_{i=1}^{s} r_i \leq L(m - t)$$

and therefore to

$$L'm' - L't'' + \sum_{i=1}^{s} r_i \leq Lm - Lt.$$

But, since $L'm' = L'L\mu = Lm$, the last inequality is equivalent to

$$L't'' - \sum_{i=1}^{s} r_i \geq Lt.$$

Hence, if t'' is such that for all $d'_1, \ldots, d'_s \in \mathbb{N}_0$ with $d'_1 + \cdots + d'_s = m' - t''$ we have

$$L't'' - \sum_{i=1}^{s} r_i \geq Lt,$$

then $\mathcal{P}$ is a (t'', m', s)-net in base $c^{L'}$. That is, we can set

$$t' = \min\{t'' \,:\, L't'' - M(t'') \geq Lt\}, \tag{4.2}$$

where

$$M(t'') = \max\left\{\sum_{i=1}^{s}(-L'd'_i \pmod L) \,:\, d'_1, \ldots, d'_s \in \mathbb{N}_0 \text{ and } \sum_{i=1}^{s} d'_i = m' - t''\right\}.$$

In the following, in order to obtain the desired estimate for t', we estimate $M(t'')$ in two different ways.

1. First we have

$$\begin{aligned} M(t'') &= \max\left\{\sum_{i=1}^{s}(-L'd'_i \pmod L) \,:\, d'_1, \ldots, d'_s \in \mathbb{N}_0 \right. \\ &\qquad\qquad \left. \text{and } \sum_{i=1}^{s} d'_i = m' - t''\right\} \\ &\leq \max\left\{\sum_{i=1}^{s}(-L' \pmod L)d'_i \,:\, d'_1, \ldots, d'_s \in \mathbb{N}_0 \right. \\ &\qquad\qquad \left. \text{and } \sum_{i=1}^{s} d'_i = m' - t''\right\} \\ &= (-L' \pmod L)(m' - t''). \end{aligned} \tag{4.3}$$

Now from (4.2) it follows that

$$t' \leq \min\{t'' \,:\, L't'' - (-L' \pmod L)(m' - t'') \geq Lt\},$$

which is satisfied for all t'' with

$$t'' \geq \left\lceil \frac{Lt + \mu L(-L' \pmod L)}{L' + (-L' \pmod L)} \right\rceil .$$

2. Further, if we define

$$N(t'') := \max \left\{ \sum_{i=1}^{s} (L'k_i \pmod L) : \right.$$
$$\left. k_i \in \{0, \ldots, L-1\} \text{ and } \sum_{i=1}^{s} k_i \equiv t'' \pmod L \right\},$$

then $M(t'') \leq N(t'')$ always.

For $N(t'')$, by its definition, we conclude the properties

$$N(t'') \leq \sum_{i=1}^{s} (L-1) = s(L-1) \tag{4.4}$$

and

$$N(t'') = L't'' + kL \tag{4.5}$$

for some integer k.

Take now $k_1, \ldots, k_{s-1}$ such that $L'k_i \equiv L-1 \pmod L$ for $1 \leq i < s$ and k_s such that $\sum_{i=1}^{s} k_i \equiv t'' \pmod L$, that is, $L'k_s \equiv L't'' + s - 1 \pmod L$. Then

$$\sum_{i=1}^{s} (L'k_i \pmod L) = (s-1)(L-1) + (L't'' + s - 1 \pmod L).$$

If $N(t'')$ were larger than the right hand side of the above equation, then, by (4.5), we had

$$N(t'') \geq (s-1)(L-1) + L > s(L-1),$$

which contradicts (4.4). Therefore we get as second estimate for $M(t'')$ that

$$M(t'') \leq (s-1)(L-1) + (L't'' + s - 1 \pmod L).$$

Again from (4.2) it follows that

$$t' \leq \min\{t'' : L't'' - (s-1)(L-1) - (L't'' + s - 1 \pmod L) \geq Lt\}.$$

We show now that the smallest t'' satisfying the second condition is

$$t'' = \left\lceil \frac{Lt + (s-1)(L-1)}{L'} \right\rceil ,$$

which implies the statement of the theorem.

Let $L't'' = Lt + (s-1)(L-1) + F$ for some integer F. Then

$$L't'' - (s-1)(L-1) - (L't'' + s - 1 \pmod L) = Lt + F - (F \pmod L)$$

and this is $\geq Lt$ if and only if $F \geq 0$. Hence the minimal t'' is given by

$$\left\lceil \frac{Lt + (s-1)(L-1)}{L'} \right\rceil,$$

and the result follows. □

4.3 $(\mathbf{T}, s)$- and (t, s)-sequences and their basic properties

A disadvantage of nets in base b is that the number of points is restricted to a power of b. At first glance one could argue that we can always choose b arbitrarily large and $m = 1$, which would mean that there is no restriction at all. However, it is intuitively obvious (and is supported by the discrepancy estimates in Chapter 5) that the structure of a (t, m, s)-net in base b becomes strong only if m is large compared with b. Hence, for a given number N of points, it is sometimes better to realize the point set with a small base b, i.e., a larger value for m, and with a suboptimal quality parameter t, than to choose a large base b (e.g. $b = N$) (and therefore a small m, for instance $m = 1$), in order to obtain an optimal quality parameter t (e.g. $t = 0$). (As we have already seen in Section 4.2 the $(0, 1, s)$-net in base b given by the points $\boldsymbol{x}_n = (n/b, \ldots, n/b)$ with $n = 0, \ldots, b-1$ has no favourable distribution property at all.)

To overcome this problem, i.e., in order to obtain net-like point sets of high distribution quality for any given number N of points, the following principal idea was born.

Try to patch up a whole infinite sequence $(\boldsymbol{x}_n)_{n \geq 0}$ in $[0, 1)^s$ from $(0, m, s)$-nets in a given base b, in the sense that for any $m \geq 1$, any subsequence of the form $\boldsymbol{x}_n, \ldots, \boldsymbol{x}_{n+b^m-1}$ of length b^m is a $(0, m, s)$-net in base b.

Such a sequence intuitively would show outstanding distribution properties. However, this demand certainly cannot be satisfied in general because of two reasons. The first reason is obvious, $(0, m, s)$-nets in a base b do not exist for all s and m. We may however replace in the property above "$(0, m, s)$-net" by "(t, m, s)-net with t as small as possible". The second reason is, in general, it is not possible to obtain the (nontrivial) net-property for all blocks of length b^m, because there is too much interference between overlapping blocks. This can be illustrated by the following example.

Example 4.27 Try to construct four points $\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$ in $[0,1)^2$ such that they form a $(0,2,2)$-net in base 2 and such that every subset of the form $\{\boldsymbol{x}_i, \boldsymbol{x}_{i+1}\}$; $i \in \{0,1,2\}$ is a $(0,1,2)$-net in base 2. For our purpose it suffices to place the $\boldsymbol{x}_i$ anywhere in the sub-cubes of the form $[A/4, (A+1)/4) \times [B/4, (B+1)/4)$, the exact place in the sub-cube is irrelevant. Without loss of generality let us start with $\boldsymbol{x}_0$ in the left lower sub-square $[0, 1/4) \times [0, 1/4)$, and note that the four points $\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$ then finally must show one of the patterns shown in Figure 4.16.

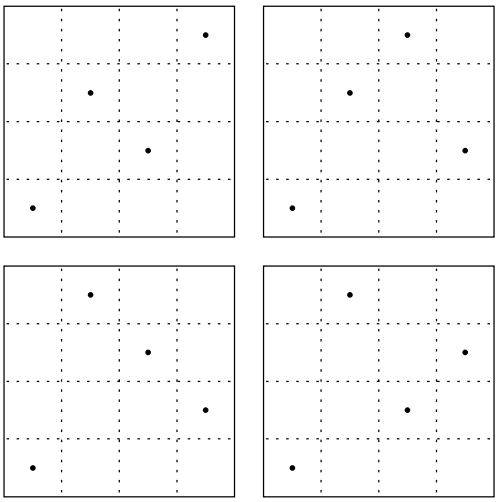

Figure 4.16 Four possible configurations for $\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$.

Two successive points must always show one of the patterns shown in Figure 4.17. Hence in the patterns of Figure 4.16 the point $\boldsymbol{x}_1$ is also pre-

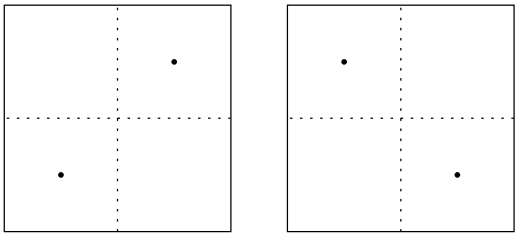

Figure 4.17 Configurations of two successive points.

scribed. However, then there is no possible choice for $\boldsymbol{x}_2$ to satisfy one of the patterns in Figure 4.17 with $\boldsymbol{x}_1, \boldsymbol{x}_2$.

Therefore we have to weaken the condition also in this aspect. We could

try this by replacing "...any sub-block of length b^m..." by "...all successive nonoverlapping sub-blocks of length b^m...".

This now leads to the definition of a (t, s)-sequence in base b.

Definition 4.28 For a given dimension $s \geq 1$, an integer base $b \geq 2$ and a nonnegative integer t, a sequence $(\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots)$ of points in $[0, 1)^s$ is called a (t, s)-*sequence in base* b if for all integers $m > t$ and $k \geq 0$ the point set consisting of the points $\boldsymbol{x}_{kb^m}, \ldots, \boldsymbol{x}_{kb^m+b^m-1}$ forms a (t, m, s)-net in base b.

Definition 4.29 A (t, s)-sequence in base b with $t \geq 1$ is called a *strict* (t, s)-*sequence in base* b if it is not a $(t-1, s)$-sequence in base b. Again we call a $(0, s)$-sequence strict by definition.

Again, we call t the *quality parameter* of the (t, s)-sequence. The notion of a (t, s)-sequence in the above form was introduced by Niederreiter in [170] for the first time. Special cases, so-called binary LP_τ-sequences, however were already investigated by Sobol′ in [251]. Another special case was introduced by Faure [66].

In [138] a generalised concept was introduced by Larcher & Niederreiter, the concept of $(\mathbf{T}, s)$-sequences in a base b.

Definition 4.30 For a given dimension $s \geq 1$, an integer base $b \geq 2$, and a function $\mathbf{T} : \mathbb{N}_0 \to \mathbb{N}_0$ with $\mathbf{T}(m) \leq m$ for all $m \in \mathbb{N}_0$, a sequence $(\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots)$ of points in $[0, 1)^s$ is called a $(\mathbf{T}, s)$-*sequence in base* b if for all integers $m \geq 0$ and $k \geq 0$ the point set consisting of the points $\boldsymbol{x}_{kb^m}, \ldots, \boldsymbol{x}_{kb^m+b^m-1}$ forms a $(\mathbf{T}(m), m, s)$-net in base b.

Definition 4.31 A $(\mathbf{T}, s)$-sequence in base b is called a *strict* $(\mathbf{T}, s)$-*sequence in base* b if for all functions $\mathbf{U} : \mathbb{N}_0 \to \mathbb{N}_0$ with $\mathbf{U}(m) \leq m$ for all $m \in \mathbb{N}_0$ and with $\mathbf{U}(m) < \mathbf{T}(m)$ for at least one $m \in \mathbb{N}_0$ it is not a $(\mathbf{U}, s)$-sequence in base b.

The concept of (t, s)-sequences in base b is contained in the concept of $(\mathbf{T}, s)$-sequences in a base b. We just have to take for $\mathbf{T}$ the constant function $\mathbf{T}(m) = t$ for all m (resp. $\mathbf{T}(m) = m$ for $m \leq t$).

By the condition $\mathbf{T}(m) \leq m$ for all m, we necessarily have $\mathbf{T}(0) = 0$, and therefore $\mathbf{T}$ is sometimes only defined for $m \geq 1$. A suitable function $\mathbf{T}$ is called a *quality function*.

If $\mathbf{T}$ is the quality function of a strict $(\mathbf{T}, s)$-sequence $(\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots)$ in base b, then for all m we have

$$\mathbf{T}(m+1) \leq \mathbf{T}(m) + 1,$$

hence the function $S(m) := m - \mathbf{T}(m)$ is monotonically increasing. This

property follows by considering a sub-block of b^{m+1} successive points of the form $\boldsymbol{x}_{kb^{m+1}}, \ldots, \boldsymbol{x}_{kb^{m+1}+b^{m+1}-1}$, and an elementary interval of order $m - \mathbf{T}(m) = (m+1) - (\mathbf{T}(m)+1)$. Since $(\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots)$ is a $(\mathbf{T}, s)$-sequence, the interval contains exactly $b^{\mathbf{T}(m)}$ points of

$$\boldsymbol{x}_{kb^{m+1}+lb^m}, \ldots, \boldsymbol{x}_{kb^{m+1}+lb^m+b^m-1}$$

for all $0 \le l \le b-1$, and so it contains exactly $b^{\mathbf{T}(m)+1}$ of the elements of

$$\boldsymbol{x}_{kb^{m+1}}, \ldots, \boldsymbol{x}_{kb^{m+1}+b^{m+1}-1}.$$

Therefore any sub-block of length b^{m+1} is fair with respect to intervals of order $(m+1) - (\mathbf{T}(m)+1)$ and consequently $\mathbf{T}(m+1) \le \mathbf{T}(m) + 1$.

What does the (t, s)-sequence property in base b mean for the first N elements of a (t, s)-sequence $(\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots)$?

1. Trivially, by definition, if N is a power of b, say $N = b^m$ for some positive integer m, then $\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$ is a (t, m, s)-net in base b. That means, in any elementary b-adic interval of volume $\frac{b^t}{N}$ there are exactly b^t points of the point set.
2. If N is a multiple of a power of b, say $N = kb^m$ for some positive integers $m \ge t$ and k, then $\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$ is a combination of k (t, m, s)-nets in base b. That means, in any elementary b-adic interval of volume $\frac{b^t}{N}$, there are exactly kb^t points of the point set.
3. In general, represent N in base b, say

   $$N = a_m b^m + a_{m-1} b^{m-1} + \cdots + a_1 b + a_0$$

   with $a_i \in \{0, 1, \ldots, b-1\}$ for $0 \le i \le m$. Then $\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$ is a combination of

a_m	(t, m, s)-nets in base b and
a_{m-1}	$(t, m-1, s)$-nets in base b and
$\vdots$	$\vdots$
a_{t+1}	$(t, t+1, s)$-nets in base b

   and further $a_0 + a_1 b + \cdots + a_t b^t$ points without a special prescribed structure. That is, if the quality parameter t is small, then $\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$ is a superposition of large point sets with strong distribution properties, smaller point sets with less restrictive distribution properties and small point sets without any prescribed distribution properties. This framework of (t, s)-sequences is the basis for the derivation of the discrepancy estimates for (t, s)-sequences, which are presented in Chapter 5.

The more general concept of $(\mathbf{T}, s)$-sequences in base b was introduced for two reasons.

1. Firstly, a quality function $\mathbf{T}$ is a more sensitive measure then a quality parameter t. For instance, a (t, s)-sequence in base b may be a strict (t, s)-sequence, (i.e., t cannot be replaced by $t-1$), but if we consider it as a $(\mathbf{T}, s)$-sequence in base b, with $\mathbf{T}(m) = t$ for all $m \geq t$, it does not have to be strict (i.e. $\mathbf{T}(m) = t$ for some $m \geq t$ can be replaced by $\mathbf{T}(m) = t - 1$ and by even smaller values). Indeed, in many concrete examples of (t, s)-sequences in base b we have a quality parameter t which is obtained by theoretical considerations. If we consider these sequences as $(\mathbf{T}, s)$-sequences, then it turns out that the real (strict) quality function $\mathbf{T}(m)$ for smaller values of m is often essentially smaller than t, and only for large m, $\mathbf{T}(m)$ is approaching t.

   However, in most cases it is very difficult to obtain good estimates for the strict quality function $\mathbf{T}$ by theoretical means. Therefore the determination of the strict quality function $\mathbf{T}$ of a $(\mathbf{T}, s)$-sequence relies in most cases on computational work.
2. The second reason for introducing quality functions is of a more theoretical nature. For certain classes of sequences (especially digital sequences, see Chapter 4.4, or Kronecker type sequences, see [133]) it turned out that their average behaviour cannot be described with a constant and therefore a bounded quality parameter t, but it can be described with a quality function $\mathbf{T}$, which may be unbounded. For corresponding results see Section 4.4.

Since any (t, m, s)-net in base b with $t \leq m - 1$ is also a $(t + 1, m, s)$-net in base b, any (t, s)-sequence in base b is also a $(t + 1, s)$-sequence in base b.

Generally, any $(\mathbf{T}, s)$-sequence in base b is also a $(\mathbf{U}, s)$-sequence in base b for all quality functions $\mathbf{U}$ with $\mathbf{U}(m) \geq \mathbf{T}(m)$ for all m.

Every point set of b^m points in $[0, 1)^s$ is a (m, m, s)-net in base b. Hence, with $\mathbf{M}(m) := m$ for all m, every sequence in $[0, 1)^s$ is a $(\mathbf{M}, s)$-sequence in base b.

Distribution properties of $(\mathbf{T}, s)$*-sequences*

Consider now $(\mathbf{T}, s)$-sequences and (t, s)-sequences. We may ask now under which conditions are they uniformly distributed modulo one (see Definition 3.1). The answer to this question is given in the following theorem.

Theorem 4.32 *A strict* $(\mathbf{T}, s)$*-sequence in any base* b *is uniformly distributed modulo one if*

$$\lim_{m\to\infty} m - \mathbf{T}(m) = \infty.$$

In particular, every (t, s)*-sequence is uniformly distributed modulo one.*

Remark 4.33 Recall that $m - \mathbf{T}(m)$ is monotonically increasing.

Proof of Theorem 4.32 Let $\mathcal{S}$ be a strict $(\mathbf{T}, s)$-sequence in base b such that $\lim_{m\to\infty} m - \mathbf{T}(m) = \infty$. Further let

$$J := \prod_{i=1}^{s} [\alpha_i, \beta_i)$$

with $0 \le \alpha_i < \beta_i \le 1$, be an arbitrary subinterval of $[0, 1)^s$, and let $\varepsilon > 0$ be given. We show that

$$\left| \frac{A(J, N, \mathcal{S})}{N} - \lambda_s(J) \right| < \varepsilon$$

for all N large enough. Then the result follows (see Definition 3.1). Let $l := rs$ with $r, s \in \mathbb{N}$ be fixed such that $\frac{2s}{b^r} < \varepsilon/2$ and let m be fixed such that $m - \mathbf{T}(m) \ge l$. Let

$$\tfrac{A_i}{b^r} \le \alpha_i < \tfrac{A_i+1}{b^r} \quad \text{and} \quad \tfrac{B_i}{b^r} \le \beta_i < \tfrac{B_i+1}{b^r}$$

for $1 \le i \le s$. Then for

$$J_1 := \prod_{i=1}^{s} \left[\frac{A_i + 1}{b^r}, \frac{B_i}{b^r} \right) \quad \text{and} \quad J_2 := \prod_{i=1}^{s} \left[\frac{A_i}{b^r}, \frac{B_i + 1}{b^r} \right)$$

we have

$$J_1 \subseteq J \subseteq J_2 \subseteq [0, 1)^s,$$

both are unions of at most b^l elementary intervals of order l, and by Lemma 3.18, $\lambda_s(J_2 \setminus J_1) \le 2s/b^r$. Hence the intervals J_1, J_2 are fair with respect to subsequences of length b^m. Therefore, for all positive integers N, we have

$$\begin{aligned} A(J, N, \mathcal{S}) - N\lambda_s(J) &\le A(J_2, N, \mathcal{S}) - N\lambda_s(J_2) + N\frac{2s}{b^r} \\ &\le A(J_2, \lfloor N/b^m \rfloor b^m, \mathcal{S}) - \left\lfloor \frac{N}{b^m} \right\rfloor b^m \lambda_s(J_2) + b^m + N\frac{2s}{b^r} \\ &= b^m + N\frac{2s}{b^r} \end{aligned}$$

and

$$\begin{aligned} A(J,N,\mathcal{S}) - N\lambda_s(J) &\geq A(J_1,N,\mathcal{S}) - N\lambda_s(J_1) - N\frac{2s}{b^r} \\ &\geq A(J_1,\lceil N/b^m\rceil b^m,\mathcal{S}) - \left\lceil \frac{N}{b^m}\right\rceil b^m\lambda_s(J_1) - b^m - N\frac{2s}{b^r} \\ &= -b^m - N\frac{2s}{b^r}, \end{aligned}$$

such that

$$\left|\frac{A(J,N,\mathcal{S})}{N} - \lambda_s(J)\right| \leq \frac{b^m}{N} + \frac{2s}{b^r} < \varepsilon$$

for N large enough. Hence the result follows. □

Note that the condition is not an "if and only if"-condition, since there are uniformly distributed sequences, having no nontrivial net-property at all. For so-called digital sequences (see Section 4.4), the above sufficient condition is also necessary and hence the above result cannot be improved.

We even have well-distribution for all $(\mathbf{T}, s)$-sequences considered in Theorem 4.32. This is an important fact for many forms of applications where the sequence in use is not used from the first point on. Sometimes, for a variety of reasons, a first sub-block of the sequence is deleted (see [223]). We prove the following result, which is, for the special case $\mathbf{T}(m) = t$ for all $m \geq t$, also proved in [98, Theorem 1], but in a less elementary way.

Theorem 4.34 *A strict $(\mathbf{T}, s)$-sequence in any base b is well-distributed modulo one if*

$$\lim_{m\to\infty} m - \mathbf{T}(m) = +\infty.$$

In particular, every (t, s)-sequence is well-distributed modulo one.

Proof For an interval $B \subseteq [0,1]^s$ let now

$$A(B,k,N,\mathcal{S}) := \#\{n \in \mathbb{N}_0 \,:\, k \leq n < k+N \text{ and } \boldsymbol{x}_n \in B\}.$$

We use the notation of the proof of Theorem 4.32. We have to show that for all $\varepsilon > 0$ there is an $N(\varepsilon)$, such that $\left|\frac{A(J,k,N,\mathcal{S})}{N} - \lambda_s(J)\right| < \varepsilon$ for all k and all $N \geq N(\varepsilon)$. Choose again $l := rs$ with $r, s \in \mathbb{N}$ such that $2s/b^r < \varepsilon/2$ and m fixed such that $m - \mathbf{T}(m) \geq l$. Consider again J_1 and J_2. Note that

$$\begin{aligned} &A(J,k,N,\mathcal{S}) - N\lambda_s(J) \\ &\leq A(J_2,k,N,\mathcal{S}) - N\lambda_s(J_2) + N\frac{2s}{b^r} \end{aligned}$$

$$\begin{aligned}
&\le A(J_2, \lfloor (N+k)/b^m \rfloor b^m, \mathcal{S}) - \left\lfloor \frac{N+k}{b^m} \right\rfloor b^m \lambda_s(J_2) \\
&\quad - (A(J_2, (\lfloor k/b^m \rfloor + 1)b^m, \mathcal{S}) - (\lfloor k/b^m \rfloor + 1) b^m \lambda_s(J_2)) + 2b^m + N\frac{2s}{b^r} \\
&= 2b^m + N\frac{2s}{b^r}
\end{aligned}$$

and

$$\begin{aligned}
&A(J, k, N, \mathcal{S}) - N\lambda_s(J) \\
&\ge A(J_1, k, N, \mathcal{S}) - N\lambda_s(J_1) - N\frac{2s}{b^r} \\
&\ge A(J_1, \lceil (N+k)/b^m \rceil b^m, \mathcal{S}) - \left\lceil \frac{N+k}{b^m} \right\rceil b^m \lambda_s(J_1) \\
&\quad - (A(J_1, (\lceil k/b^m \rceil - 1)b^m, \mathcal{S}) - (\lceil k/b^m \rceil - 1) b^m \lambda_s(J_1)) - 2b^m - N\frac{2s}{b^r} \\
&= -2b^m - N\frac{2s}{b^r}.
\end{aligned}$$

Hence we obtain that

$$\left| \frac{A(J, k, N, \mathcal{S})}{N} - \lambda_s(J) \right| \le \frac{2b^m}{N} + \frac{2s}{b^r} < \varepsilon$$

for all k and all $N \ge 4b^m \varepsilon^{-1}$. Therefore the result follows. □

A first example

As a first nontrivial example let us try to artificially generate a $(\mathbf{T}, s)$-sequence in base b from regular lattices. We restrict ourselves to base $b = 2$. The points $\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots$ have the following form:

$$\begin{aligned}
&\boldsymbol{x}_0 = (0, \ldots, 0), && \boldsymbol{x}_1 = (1/2, \ldots, 1/2), \\
&\boldsymbol{x}_2 = (1/2, 0, \ldots, 0), && \boldsymbol{x}_3 = (0, 1/2, \ldots, 1/2), \\
&\quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\boldsymbol{x}_{2^s-2} = (1/2, \ldots, 1/2, 0), && \boldsymbol{x}_{2^s-1} = (0, \ldots, 0, 1/2).
\end{aligned}$$

The points are ordered such that $\boldsymbol{x}_{2j} + \boldsymbol{x}_{2j+1} = (1/2, \ldots, 1/2)$ for all $0 \le j \le 2^{s-1} - 1$. Let now $\boldsymbol{y}_j^{(k)} := \boldsymbol{x}_j / 2^{k-1}$ for all $0 \le j \le 2^s - 1$, i.e.,

$$\begin{aligned}
&\boldsymbol{y}_0^{(k)} = (0, \ldots, 0), && \boldsymbol{y}_1^{(k)} = (1/2^k, \ldots, 1/2^k), \\
&\boldsymbol{y}_2^{(k)} = (1/2^k, 0, \ldots, 0), && \boldsymbol{y}_3^{(k)} = (0, 1/2^k, \ldots, 1/2^k), \\
&\quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\boldsymbol{y}_{2^s-2}^{(k)} = (1/2^k, \ldots, 1/2^k, 0), && \boldsymbol{y}_{2^s-1}^{(k)} = (0, \ldots, 0, 1/2^k).
\end{aligned}$$

If we have already constructed the points $\boldsymbol{x}_0, \dots, \boldsymbol{x}_{2^{sk}-1}$, we get the following $2^{s(k+1)} - 2^{sk}$ points by:

$$
\begin{array}{lll}
\boldsymbol{x}_{2^{sk}} := \boldsymbol{x}_0 + \boldsymbol{y}_1^{(k+1)}, & \dots, & \boldsymbol{x}_{2\cdot 2^{sk}-1} := \boldsymbol{x}_{2^{sk}-1} + \boldsymbol{y}_1^{(k+1)}, \\
\boldsymbol{x}_{2\cdot 2^{sk}} := \boldsymbol{x}_0 + \boldsymbol{y}_2^{(k+1)}, & \dots, & \boldsymbol{x}_{3\cdot 2^{sk}-1} := \boldsymbol{x}_{2^{sk}-1} + \boldsymbol{y}_2^{(k+1)}, \\
\dots\dots\dots & & \\
\boldsymbol{x}_{(2^s-1)\cdot 2^{sk}} := \boldsymbol{x}_0 + \boldsymbol{y}_{2^s-1}^{(k+1)}, & \dots, & \boldsymbol{x}_{2^{s(k+1)}-1} := \boldsymbol{x}_{2^{sk}-1} + \boldsymbol{y}_{2^s-1}^{(k+1)},
\end{array}
$$

for all $k \in \mathbb{N}$.

We illustrate the generation procedure in dimension 2. We start with the four points $\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$ as indicated in Figure 4.18.

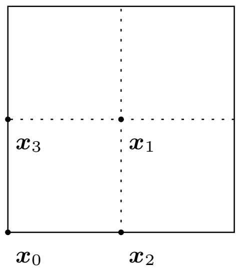

Figure 4.18 The four start points $\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$.

This four-scheme always occurs in the subsequent procedure. Then we repeat this four-scheme consecutively in the four sub-squares according to their numbering by $\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$, see Figure 4.19.

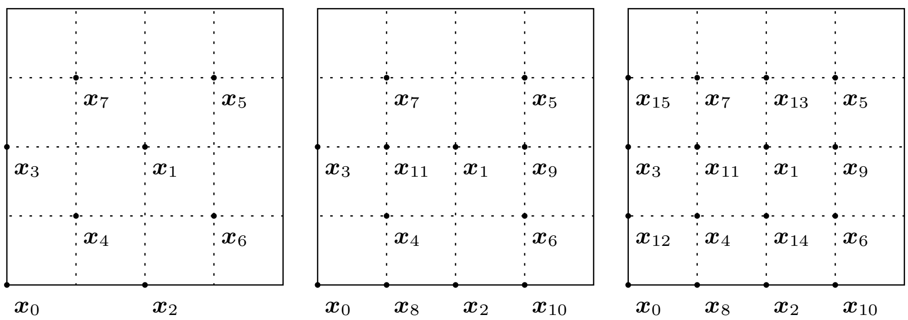

Figure 4.19 Construction of $\boldsymbol{x}_0, \dots, \boldsymbol{x}_{15}$.

Then we repeat this four-scheme consecutively in the 4^2 sub-squares according to their numbering by $\boldsymbol{x}_0, \dots, \boldsymbol{x}_{15}$, and we get Figure 4.20.

For $s = 1$ we obtain the van der Corput sequence in base 2 (see Definition 3.10) in this way. We have the following result.

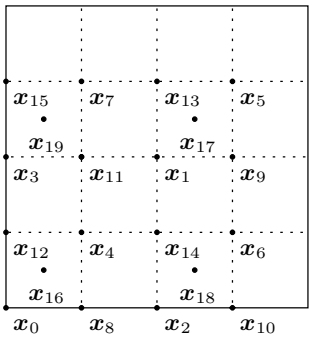

Figure 4.20 Construction of $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{19}$.

Proposition 4.35 *The above generated sequence* $(\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots)$ *in* $[0,1)^s$ *is a strict* $(\mathbf{T}, s)$*-sequence in base* 2 *with* $\mathbf{T}(m) = m - \left\lceil \frac{m}{s} \right\rceil$. *In particular, the van der Corput sequence in base* 2 *is a* $(0,1)$*-sequence in base 2.*

Proof For $k \in \mathbb{N}$ let $\mathcal{S}_k$ denote the first 2^{sk} points of the sequence, hence

$$\mathcal{S}_k = \left\{ \left(\frac{a_1}{2^k}, \ldots, \frac{a_s}{2^k} \right) \,:\, 0 \le a_i < 2^k \right\}.$$

The first 2^{sk} points of the sequence are fair with respect to any elementary interval of order k, because of the net property of the regular lattice shown in Corollary 4.11. Now we show the fairness of a point set $\mathcal{P}$ of the form

$$\mathcal{P} = \{\boldsymbol{x}_{p2^{sk+j}}, \ldots, \boldsymbol{x}_{p2^{sk+j}+2^{sk+j}-1}\}, \tag{4.6}$$

for arbitrarily chosen $p \in \mathbb{N}$ and $j \in \mathbb{N}_0$, with respect to all 2-adic elementary intervals of order $k+1$.

The case $j = 0$ follows from the above considerations concerning $\mathcal{S}_k$. Now let $j \in \mathbb{N}$. By the construction method for the sequence $(\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots)$ it follows that there are $\boldsymbol{z}_0, \ldots, \boldsymbol{z}_{2^j-1}$, such that

$$\mathcal{P} = \{\boldsymbol{x} + \boldsymbol{z}_q \,:\, \boldsymbol{x} \in \mathcal{S}_k \text{ and } 0 \le q < 2^j\}$$

and for $\boldsymbol{z}_q = (z_{q,1}, \ldots, z_{q,s})$ we can assume without loss of generality that

$$0 \le z_{q,i} < \frac{1}{2^k}$$

for $1 \le i \le s$, $0 \le q < 2^j$ and

$$|z_{2r+1,i} - z_{2r,i}| = \frac{1}{2^{k+1}}$$

for $1 \le i \le s$, $0 \le r \le 2^{j-1} - 1$. Let

$$J = \prod_{i=1}^{s} \left[\frac{A_i}{2^{d_i}}, \frac{A_i + 1}{2^{d_i}} \right),$$

where $d_1, \ldots, d_s \in \mathbb{N}_0$, $\sum_{i=1}^{s} d_i = k + 1$, and $0 \le A_i < 2^{d_i}$ for all $1 \le i \le s$.

Firstly we consider the case where $d_1, \ldots, d_s \le k$. For $\boldsymbol{x} = \left(x_1/2^k, \ldots, x_s/2^k\right) \in J$, where $x_1, \ldots, x_s$ are integers, we have that also $\boldsymbol{x} + \boldsymbol{z}_q \in J$. This holds since

$$A_i 2^{k-d_i} \le x_i < (A_i + 1)2^{k-d_i}$$

and $0 \le z_{q,i} 2^k < 1$ imply that

$$A_i 2^{k-d_i} \le x_i + z_{q,i} 2^k < (A_i + 1)2^{k-d_i}.$$

Hence there are exactly $2^{sk-k-1+j}$ points of $\mathcal{P}$ in J. (Recall that any interval of the form $\prod_{i=1}^{s} [B/2^k, (B+1)/2^k)$ contains exactly one point of $\mathcal{S}_k$.)

For the second case we can assume without loss of generality that $d_1 = k + 1$ and $d_2 = \cdots = d_s = 0$. For $\boldsymbol{x} = \left(x_1/2^k, \ldots, x_s/2^k\right)$, where $x_1, \ldots, x_s$ are integers, with $x_1 = \lfloor A_1/2 \rfloor$ and $0 \le x_i < 2^k$ for $2 \le i \le s$, we have that $\boldsymbol{x} + \boldsymbol{z}_q \in J$ if and only if $\boldsymbol{z}_q$ is of the following form

$$z_{q,1} < \frac{1}{2^{k+1}}$$

if A_1 is even, or

$$z_{q,1} \ge \frac{1}{2^{k+1}}$$

if A_1 is odd.

In either case there are 2^{j-1} elements $\boldsymbol{z}_q$ with $\boldsymbol{x} + \boldsymbol{z}_q \in J$, so altogether we have $2^{sk-k+j-1}$ points of $\mathcal{P}$ in J.

These two cases show that the point set $\mathcal{P}$ from (4.6) is a $(sk + j - k - 1, sk + j, s)$-net in base 2 for all p and j. If we write m as $m = sk + j$, then we find that a point set of the form $\{\boldsymbol{x}_{p2^m}, \ldots, \boldsymbol{x}_{p2^m + 2^m - 1}\}$ is, for all p and m, a $\left(m - \left\lceil \frac{m}{s} \right\rceil, m, s\right)$-net in base 2.

To prove strictness, it suffices to show that the point set $\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{2^m - 1}\}$ is a strict $\left(m - \left\lceil \frac{m}{s} \right\rceil, m, s\right)$-net in base 2 for all m. Choose any $m = sk + j$ and consider the elementary interval J of order $k + 2$,

$$J = \left[0, \frac{1}{2^{k+2}}\right) \times \prod_{i=1}^{s} [0, 1).$$

If $\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{2^m - 1}\}$ would be a $\left(m - \left\lceil \frac{m}{s} \right\rceil - 1, m, s\right)$-net in base 2, then there should be $2^{sk-k+j-2}$ points of the point set in J. But for the first components

$x_{n,1}$ of $\boldsymbol{x}_n \in J$, $0 \le n \le 2^{sk} - 1$, we have $0 \le x_{n,1} < 1/4$. Hence there are $2^{(s-1)k}$ points from $\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{2^{sk}-1}\}$ in J and from the construction method for the $\boldsymbol{x}_n$ it follows that there are even $2^{sk-k+j-1}$ points from $\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{2^{sk+j}-1}\}$ in J. As $2^{sk-k+j-1} > 2^{sk-k+j-2}$ we obtain a contradiction and hence the result follows. □

Further examples of $(\mathbf{T}, s)$-sequences and (t, s)-sequences are given in the subsequent section and in Chapter 8.

Existence of $(0, s)$*-sequences in base* b

As for $(0, m, s)$-nets in base b it is clear that a $(0, s)$-sequence in base b cannot exist for all dimensions s. We have the following result.

Corollary 4.36 *A $(0, s)$-sequence in base b cannot exist if $s \ge b + 1$.*

This corollary is a consequence of Corollary 4.19 in Section 4.2 and of the following corollary (see [175, Lemma 4.22]).

Corollary 4.37 *If there exists a (t, s)-sequence in base b, then, for every $m \ge t$, there exists a $(t, m, s + 1)$-net in base b.*

And this again is a consequence of the following lemma.

Lemma 4.38 *Let $(\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots)$ be a $(\mathbf{T}, s)$-sequence in base b. Then, for every m, the point set $\{\boldsymbol{y}_0, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_{b^m-1}\}$ with $\boldsymbol{y}_k := (k/b^m, \boldsymbol{x}_k)$, $0 \le k < b^m$, is an $(r(m), m, s+1)$-net in base b with $r(m) := \max\{\mathbf{T}(0), \ldots, \mathbf{T}(m)\}$.*

Proof Let $J = \prod_{i=1}^{s+1} \left[\frac{A_i}{b^{d_i}}, \frac{A_i+1}{b^{d_i}}\right)$ be an elementary interval of order $m - r(m)$. Then $\boldsymbol{y}_k \in J$ if and only if

$$\frac{k}{b^m} \in \left[\frac{A_1}{b^{d_1}}, \frac{A_1+1}{b^{d_1}}\right) \quad \text{and} \quad \boldsymbol{x}_k \in \prod_{i=2}^{s+1} \left[\frac{A_i}{b^{d_i}}, \frac{A_i+1}{b^{d_i}}\right).$$

The first condition leads to

$$A_1 b^{m-d_1} \le k < A_1 b^{m-d_1} + b^{m-d_1}.$$

Since $(\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots)$ is a $(\mathbf{T}, s)$-sequence in base b, the points $\boldsymbol{x}_{A_1 b^{m-d_1}+l}$, $0 \le l \le b^{m-d_1} - 1$, form an $(r(m), m-d_1, s)$-net in base b, because $r(m) \ge \mathbf{T}(m - d_1)$. The interval $\prod_{i=2}^{s+1} \left[A_i/b^{d_i}, (A_i+1)/b^{d_i}\right)$ has volume $b^{-d_2-\cdots-d_{s+1}} = b^{-m+d_1+r(m)}$ and therefore contains exactly $b^{r(m)}$ of the points $\boldsymbol{x}_{A_1 b^{m-d_1}+l}$, $0 \le l \le b^{m-d_1} - 1$. Consequently J contains exactly $b^{r(m)}$ of the points $\boldsymbol{y}_k$, $0 \le k \le b^m - 1$, and the result follows. □

Example 4.39 Let $(x_0, x_1, \ldots)$ be the van der Corput sequence in base b, which is an example of a $(0,1)$-sequence in base b. Then the point set $\{\boldsymbol{y}_0, \ldots, \boldsymbol{y}_{b^m-1}\}$, where $\boldsymbol{y}_k := (k/b^m, x_k)$ for $0 \le k \le b^m - 1$, is the Hammersley point set in base b with b^m points and hence a $(0, m, 2)$-net in base b (see Figure 4.21).

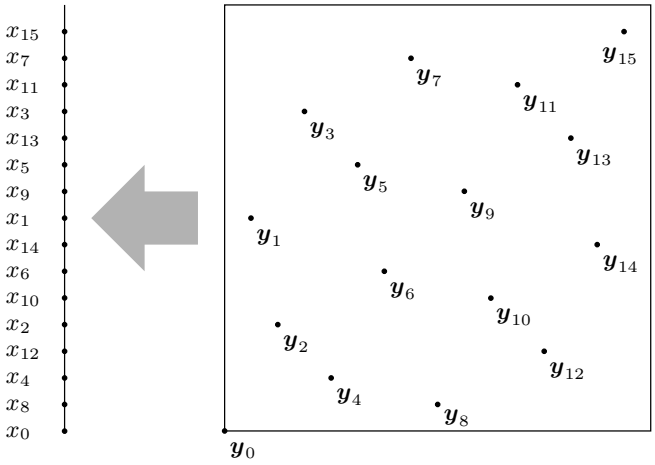

Figure 4.21 Hammersley point set in base 2 with 16 points. The projection to the second coordinate gives the first 16 elements of the van der Corput sequence in base 2.

The advantage of (t, s)-sequences is that also subsequences show favourable distribution properties (and even net properties).

Faure- and Niederreiter-sequences provide, for every prime power base b and any $s \le b$, a $(0, s)$-sequence in base b, see Chapter 8. Consequently we obtain the following result.

Corollary 4.40 *A $(0, s)$-sequence in a prime power base b exists if and only if $s \le b$.*

Again it is more difficult to give sharp existence results for $(0, s)$-sequences in composite bases b. As a singular result from Lemma 4.21 in Section 4.2 and Corollary 4.37 it follows that:

Corollary 4.41 *There does not exist a $(0, 3)$-sequence in base 6.*

Propagation rules for sequences

Base propagation rules for (t, s)- and $(\mathbf{T}, s)$-sequences can be transferred from the corresponding rules for (t, m, s)-nets. Thus, as a consequence from Corollary 4.23 in Section 4.2, we obtain the following result.

Corollary 4.42 *Any $(\mathbf{T}, s)$-sequence $(\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots)$ in base b is a $(\mathbf{U}, s)$-sequence in base b^k, where*

$$\mathbf{U}(m) := \left\lceil \frac{\mathbf{T}(km)}{k} \right\rceil .$$

Proof Take any subsequence of the form

$$(\boldsymbol{x}_{l(b^k)^m}, \boldsymbol{x}_{l(b^k)^m+1}, \ldots, \boldsymbol{x}_{l(b^k)^m+(b^k)^m-1}),$$

where $l \geq 0$, of the sequence $(\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots)$ which is a $(\mathbf{T}, s)$-sequence in base b. Then the elements of this subsequence form a $(\mathbf{T}(km), km, s)$-net in base b and therefore, by Corollary 4.23 in Section 4.2, a $(\lceil \mathbf{T}(km)/k \rceil, m, s)$-net in base b^k. The result follows by the definition of a $(\mathbf{U}, s)$-sequence in base b^k. □

We also have the following result, see [190, Proposition 5].

Corollary 4.43 *Any (t, s)-sequence in base b is a $(\lceil t/k \rceil, s)$-sequence in base b^k.*

Again a similar converse assertion does not hold in general. As a counterexample serves, for instance, the van der Corput sequence in base 4. It is a $(0, 1)$-sequence in base 4, but certainly not a $(0, 1)$-sequence in base 2. This assertion can be shown by observing that for any integer $k \geq 1$ the two points $x_{4^k} = \frac{1}{4^{k+1}}$ and $x_{4^k+1} = \frac{1}{4} + \frac{1}{4^{k+1}}$ are both contained in $[0, \frac{1}{2})$. Hence they do not form a $(0, 1, 1)$-net in base 2 and therefore $(x_0, x_1, \ldots)$ is not a $(0, 1)$-sequence in base 2.

However, we can use Theorem 4.24 from Section 4.2 to obtain a base change result for $(\mathbf{T}, s)$- and (t, s)-sequences.

Theorem 4.44 *For given integers $c \geq 2$, L and $L' \geq 1$ with $\gcd(L, L') = 1$ we have that every $(\mathbf{T}, s)$-sequence $(\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots)$ in base c^L is a $(\mathbf{U}, s)$-sequence in base $c^{L'}$, where*

$$\mathbf{U}(m) = m \pmod{L} + \min(V(m), W(m)),$$

with

$$V(m) := \left\lceil \frac{L\mathbf{T}(L'\lfloor m/L \rfloor) + \lfloor m/L \rfloor L((-L') \pmod{L})}{L' + ((-L') \pmod{L})} \right\rceil$$

and

$$W(m) := \left\lceil \frac{L\mathbf{T}(L'\lfloor m/L \rfloor) + (s-1)(L-1)}{L'} \right\rceil .$$

Before we prove the theorem let us again consider the special cases $L = 1$, $L' = 1$ and $\mathbf{T} \equiv t$. For $L = 1$ we obtain $\mathbf{U}(m) = \left\lceil \frac{\mathbf{T}(L'm)}{L'} \right\rceil$ and therefore again Corollary 4.42. For $L' = 1$ we obtain the following corollary, which is a generalisation of [190, Proposition 4].

Corollary 4.45 *For a given base $b \geq 2$ and any integer $L \geq 1$, every $(\mathbf{T}, s)$-sequence in base b^L is a $(\mathbf{U}, s)$-sequence in base b with*

$$\mathbf{U}(m) = m \pmod{L} + \min(V(m), W(m))$$

with

$$V(m) := \mathbf{T}(\lfloor m/L \rfloor) + \lfloor m/L \rfloor (L-1)$$

and

$$W(m) := L\mathbf{T}(\lfloor m/L \rfloor) + (s-1)(L-1).$$

For $\mathbf{T} \equiv t$ we obtain the following result, see [190, Proposition 4].

Corollary 4.46 *For given integers $c \geq 2$, L and $L' \geq 1$ with $\gcd(L, L') = 1$ we have that every (t, s)-sequence in base c^L is a (n, s)-sequence in base $c^{L'}$, where*

$$n = L - 1 + \left\lceil \frac{Lt + (s-1)(L-1)}{L'} \right\rceil .$$

Proof of Theorem 4.44 Consider a subsequence of $(c^{L'})^m$ elements of the sequence $(\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots)$ of the form

$$(\boldsymbol{x}_{k(c^{L'})^m}, \ldots, \boldsymbol{x}_{k(c^{L'})^m + (c^{L'})^m - 1}).$$

Represent m in the form $m = pL + r$ with $0 \leq r < L$, then the above subsequence is

$$(\boldsymbol{x}_{kc^{rL'}(c^L)^{pL'}}, \ldots, \boldsymbol{x}_{kc^{rL'}(c^L)^{pL'} + c^{rL'}(c^L)^{pL'} - 1})$$

and so a multiset-union of $(c^{L'})^r$ subsequences of length $(c^L)^{pL'}$. Any such subsequence, by the $(\mathbf{T}, s)$-sequence property in base c^L of $(\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots)$, forms a $(\mathbf{T}(pL'), pL', s)$-net in base c^L and therefore by Theorem 4.24 in Section 4.2 (note that $p = \lfloor m/L \rfloor$) is a (q, pL, s)-net in base $c^{L'}$ with $q = \min(v, w)$, where

$$v = \left\lceil \frac{L\,\mathbf{T}(L'\lfloor m/L \rfloor) + \lfloor m/L \rfloor L((-L') \pmod{L})}{L' + ((-L') \pmod{L})} \right\rceil$$

and

$$w = \left\lceil \frac{L\,\mathbf{T}(L'\lfloor m/L \rfloor) + (s-1)(L-1)}{L'} \right\rceil .$$

By Lemma 4.14, concerning the combination of (t, m, s)-nets, now the original sub-block as a combination of $(c^{L'})^r$ (q, pL, s)-nets in base $c^{L'}$ forms a $(q+r, pL+r, s) = (q+r, m, s)$-net in base $c^{L'}$. Note that $r \equiv m \pmod{L}$. Hence the result follows by the definition of a $(\mathbf{U}, s)$-sequence in base $c^{L'}$. □

4.4 Digital (t, m, s)-nets and digital $(\mathbf{T}, s)$- and (t, s)-sequences

The concept of digital (t, m, s)-nets, and digital $(\mathbf{T}, s)$- and (t, s)-sequences is a general framework for the construction of (t, m, s)-nets and $(\mathbf{T}, s)$- and (t, s)-sequences. In fact, until now, essentially all concrete (t, m, s)-nets, and $(\mathbf{T}, s)$- and (t, s)-sequences which are of relevance for applications are digital (t, m, s)-nets, and digital $(\mathbf{T}, s)$- and (t, s)-sequences. For short, in the following we speak of digital point sets. In particular, all relevant examples provided so far can be introduced in terms of digital point sets.

Using the framework of digital point sets, allows us

1. to provide the (t, m, s)-net, the $(\mathbf{T}, s)$- or (t, s)-sequence in an easy way (in the form of s matrices);
2. to determine the quality parameter t or $\mathbf{T}$ in a rather fast way;
3. to describe the properties of point sets in question in terms of properties of the matrices mentioned above, i.e., the search for point sets of high quality can be restricted to the search for matrices with certain properties.

Although one can introduce digital nets in arbitrary integer bases $b \geq 2$, we restrict ourselves to *prime power* bases b only in the following. The main motivation for this restriction is that there exists a finite field of order b if and only if b is a prime power. This leads a simpler and clearer construction of digital point sets. Some points of the analysis of digital nets in arbitrarily chosen bases are much more involved compared with the prime power base case, where the construction principle is much simpler. Furthermore, the prime power base case (and even the prime base case) is also for practical applications the most important one.

Most of the results and ideas which we give below can be generalised to digital nets in arbitrary integer bases $b \geq 2$. For a detailed treatment of the general case we refer to [139, Section 1] and [175, Section 4].

Digital (t, m, s)*-nets*

To construct a digital (t, m, s)-net in a prime power base b, we use the finite field $\mathbb{F}_b$ with b elements and a bijection $\varphi : \{0, \ldots, b-1\} \to \mathbb{F}_b$ with $\varphi(0) = \overline{0}$, the neutral element of addition in $\mathbb{F}_b$. We speak then of a "digital (t, m, s)-net over the field $\mathbb{F}_b$" instead of "digital (t, m, s)-net in base b". (For arbitrary b one has to choose a finite commutative Ring R with identity of order b, see [135, 139, 175] for more information.)

The elements of $\mathbb{F}_b$ are denoted by $\bar{0}, \bar{1}, \ldots, \overline{b-1}$ respectively and we use the bijection $\varphi(j) := \overline{j}$ for $j \in \{0, \ldots, b-1\}$. If b is a prime, then we identify $\mathbb{F}_b$ with $\mathbb{Z}_b$, the set of residue classes modulo b with addition and multiplication modulo b, which in turn we identify with the elements of $\{0, \ldots, b-1\}$. Therefore we omit the bijection φ and the bar in this case.

Let us explain the concept of digital (t, m, s)-nets over $\mathbb{F}_b$. That is, we want to construct a (t, m, s)-net $\{\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{b^m-1}\}$ in base b by the digital method. To generate such a point set we first have to choose $m \times m$ matrices $C_1, \ldots, C_s$ (one for each component) over $\mathbb{F}_b$, that is, with entries from $\mathbb{F}_b$. For example, to generate a $(t, 4, 2)$-net over $\mathbb{Z}_2$ take the matrices

$$C_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad C_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{4.7}$$

To generate now one of the points $\boldsymbol{x}_n = (x_{n,1}, \ldots, x_{n,s})$, with $0 \le n < b^m$, of the net, we first write n in its b-adic (i.e. base b) expansion $n = \sum_{j=0}^{m-1} a_j b^j$, with digits $a_j \in \{0, \ldots, b-1\}$. Note that $0 \le n < b^m$ and therefore it suffices to consider only j with $0 \le j \le m-1$. Then take the m-dimensional column vector

$$\mathbf{n} := \begin{pmatrix} \varphi(a_0) \\ \varphi(a_1) \\ \vdots \\ \varphi(a_{m-1}) \end{pmatrix} \in (\mathbb{F}_b^m)^\top.$$

For example, to generate the point $\boldsymbol{x}_{11} = (x_{11,1}, x_{11,2})$ of the $(t, 4, 2)$-net over $\mathbb{Z}_2$ from above, write

$$11 = 1 \cdot 2^0 + 1 \cdot 2^1 + 0 \cdot 2^2 + 1 \cdot 2^3,$$

which corresponds to the vector

$$\mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

or to generate the point $\boldsymbol{x}_7 = (x_{7,1}, x_{7,2})$, write

$$7 = 1 \cdot 2^0 + 1 \cdot 2^1 + 1 \cdot 2^2 + 0 \cdot 2^3,$$

which corresponds to the vector

$$\mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

To generate the point $\boldsymbol{x}_n = (x_{n,1}, \ldots, x_{n,s})$ we explain how to generate the ith coordinate:

The ith coordinate $x_{n,i}$ is obtained by multiplying the ith matrix C_i by $\mathbf{n}$ over $\mathbb{F}_b$, which gives as result an m-dimensional vector of elements of $\mathbb{F}_b$, say

$$C_i \mathbf{n} = \begin{pmatrix} \overline{y}_{n,i,1} \\ \vdots \\ \overline{y}_{n,i,m} \end{pmatrix} \in (\mathbb{F}_b^m)^\top.$$

The elements $\varphi^{-1}(\overline{y}_{n,i,j}) \in \{0, \ldots, b-1\}$ are now the b-adic digits of $x_{n,i}$, i.e.,

$$x_{n,i} = \frac{\varphi^{-1}(\overline{y}_{n,i,1})}{b} + \frac{\varphi^{-1}(\overline{y}_{n,i,2})}{b^2} + \cdots + \frac{\varphi^{-1}(\overline{y}_{n,i,m})}{b^m}.$$

Definition 4.47 We call the point set $\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{b^m-1}\}$ constructed as introduced above a *digital net over $\mathbb{F}_b$ with generating matrices $C_1, \ldots, C_s$* or short a *digital net*.

Since any point set consisting of b^m points in $[0,1)^s$ is a (t, m, s)-net in base b with a certain quality parameter t we also speak of a *digital (t, m, s)-net over $\mathbb{F}_b$*.

Therefore, to provide the b^m points in dimension s, it suffices to provide s matrices of size $m \times m$ over $\mathbb{F}_b$. This of course simplifies storage of the point sets.

As already mentioned, in most cases the finite field $\mathbb{Z}_b$ with b prime is chosen for practical applications, and indeed $\mathbb{Z}_2$ is the most frequent choice.

We remark again that in this case we can omit the bijection φ as we identify $\mathbb{Z}_b$ with the elements $\{0, \ldots, b-1\}$.

Example 4.48 Consider again the $(t, 4, 2)$-net over $\mathbb{Z}_2$ with generating matrices (4.7). To construct for instance $\boldsymbol{x}_{11}$ we have $n = 11$ and therefore

$$C_1\mathbf{n} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$C_2\mathbf{n} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

Hence $x_{11,1} = \frac{1}{2} + \frac{1}{4} + \frac{1}{16} = \frac{13}{16}$ and $x_{11,2} = \frac{1}{2} + \frac{1}{8} + \frac{1}{16} = \frac{11}{16}$, and thus $\boldsymbol{x}_{11} = \left(\frac{13}{16}, \frac{11}{16}\right)$.

For $n = 7$ we have

$$C_1\mathbf{n} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$C_2\mathbf{n} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Hence $x_{7,1} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$ and $x_{7,2} = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{7}{16}$, and thus $\boldsymbol{x}_7 = \left(\frac{7}{8}, \frac{7}{16}\right)$. Determining all 16 points shows that this example just gives the 16 point Hammersley point set in base 2.

Example 4.49 To illustrate the generation procedure we provide one example of a digital $(t, 3, 2)$-net over $\mathbb{F}_4$, the finite field of order 4. Let $\mathbb{F}_4 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}\}$. We identify the elements $\overline{0}, \overline{1}, \overline{2}, \overline{3}$ of $\mathbb{F}_4$ with the 4-adic digits $0, 1, 2, 3$ respectively, i.e., $\varphi(i) = \overline{i}$ for $i \in \{0, 1, 2, 3\}$. Addition and

multiplication in $\mathbb{F}_4$ are defined by the following tables:

$+$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{0}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{1}$	$\bar{1}$	$\bar{0}$	$\bar{3}$	$\bar{2}$
$\bar{2}$	$\bar{2}$	$\bar{3}$	$\bar{0}$	$\bar{1}$
$\bar{3}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	$\bar{0}$

$\cdot$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$
$\bar{1}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{2}$	$\bar{0}$	$\bar{2}$	$\bar{3}$	$\bar{1}$
$\bar{3}$	$\bar{0}$	$\bar{3}$	$\bar{1}$	$\bar{2}$

Choose the 3×3 matrices C_1 and C_2 over $\mathbb{F}_4$ by

$$C_1 = \begin{pmatrix} \bar{1} & \bar{0} & \bar{0} \\ \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{2} & \bar{2} \end{pmatrix} \quad \text{and} \quad C_2 = \begin{pmatrix} \bar{2} & \bar{3} & \bar{1} \\ \bar{0} & \bar{0} & \bar{1} \\ \bar{0} & \bar{1} & \bar{0} \end{pmatrix}.$$

To demonstrate how to generate the $4^3 = 64$ points $\boldsymbol{x}_n = (x_{n,1}, x_{n,2}) \in [0,1)^2$ let us generate $\boldsymbol{x}_{35}$. We have

$$35 = 3 \cdot 4^0 + 0 \cdot 4^1 + 2 \cdot 4^2$$

which corresponds to the vector

$$\mathbf{n} = \begin{pmatrix} \bar{3} \\ \bar{0} \\ \bar{2} \end{pmatrix} \in (\mathbb{F}_4^3)^\top.$$

Now

$$C_1 \begin{pmatrix} \bar{3} \\ \bar{0} \\ \bar{2} \end{pmatrix} = \begin{pmatrix} \bar{1} & \bar{0} & \bar{0} \\ \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{2} & \bar{2} \end{pmatrix} \begin{pmatrix} \bar{3} \\ \bar{0} \\ \bar{2} \end{pmatrix} = \begin{pmatrix} \bar{3} \\ \bar{0} \\ \bar{3} \end{pmatrix} \in (\mathbb{F}_4^3)^\top$$

and hence $x_{35,1} = \frac{3}{4} + \frac{0}{16} + \frac{3}{64} = \frac{51}{64}$. Further

$$C_2 \begin{pmatrix} \bar{3} \\ \bar{0} \\ \bar{2} \end{pmatrix} = \begin{pmatrix} \bar{2} & \bar{3} & \bar{1} \\ \bar{0} & \bar{0} & \bar{1} \\ \bar{0} & \bar{1} & \bar{0} \end{pmatrix} \begin{pmatrix} \bar{3} \\ \bar{0} \\ \bar{2} \end{pmatrix} = \begin{pmatrix} \bar{3} \\ \bar{2} \\ \bar{0} \end{pmatrix} \in (\mathbb{F}_4^3)^\top$$

and hence $x_{35,2} = \frac{3}{4} + \frac{2}{16} + \frac{0}{64} = \frac{7}{8}$. Therefore we have $\boldsymbol{x}_{35} = \left(\frac{51}{64}, \frac{7}{8}\right)$.

The quality parameter of digital nets

Trivially, since the number of points N is b^m, the resulting point set is a (t, m, s)-net in base b (every set of b^m points in $[0,1)^s$ is an (m, m, s)-net at least). But what is the real, strict quality parameter t of the point set, generated in the above way? The answer is given with the help of the following

quantity ρ which, in some sense, "measures" the "linear independence of the s matrices $C_1, \ldots, C_s$".

Definition 4.50 Let b be a prime power and let $C_1, \ldots, C_s$ be $m \times m$ matrices with entries from the finite field $\mathbb{F}_b$. Let $\rho = \rho(C_1, \ldots, C_s)$ be the largest integer such that for any choice of $d_1, \ldots, d_s \in \mathbb{N}_0$, with $d_1 + \cdots + d_s = \rho$, the following holds:

the first d_1 row vectors of C_1 together with
the first d_2 row vectors of C_2 together with
$\vdots$
the first d_s row vectors of C_s,

(these are together ρ vectors in $\mathbb{F}_b^m$) are linearly independent over the finite field $\mathbb{F}_b$. We call ρ the *linear independence parameter* of the matrices $C_1, \ldots, C_s$.

Example 4.51 Consider C_1, C_2 over $\mathbb{Z}_2$ from the example above,

$$C_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad C_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Clearly ρ is at most 4, since there never exist more than 4 linearly independent 4-dimensional vectors over $\mathbb{Z}_2$. However, ρ is indeed 4 in this example, since for every choice of $d_1, d_2 \geq 0$ with $d_1 + d_2 = 4$, the first d_1 rows of C_1 together with the first d_2 rows of C_2 provide the system of the 4 canonical row-vectors $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$, which are linearly independent over $\mathbb{Z}_2$.

Now we can determine the strict quality parameter t of a digital net generated by matrices $C_1, \ldots, C_s$ over $\mathbb{Z}_b$. This is a special case of [175, Theorem 4.28].

Theorem 4.52 *Let b be a prime power. The point set constructed by the digital method with the $m \times m$ matrices $C_1, \ldots, C_s$ over a finite field $\mathbb{F}_b$ is a strict $(m - \rho, m, s)$-net in base b, where $\rho = \rho(C_1, \ldots, C_s)$ is the linear independence parameter defined in Definition 4.50.*

Proof First we have to show that every elementary interval of order ρ, i.e., of volume $b^{-\rho}$ contains exactly $b^{m-\rho}$ of the generated points. Let

$$J = \prod_{i=1}^{s} \left[\frac{A_i}{b^{d_i}}, \frac{A_i + 1}{b^{d_i}} \right)$$

with $d_1,\dots,d_s \in \mathbb{N}_0$ such that $d_1+\cdots+d_s=\rho$ and $0 \le A_i < b^{d_i}$ for $1 \le i \le s$ be such an interval. We ask for which n is $\boldsymbol{x}_n = (x_{n,1},\dots,x_{n,s})$ contained in J, i.e., for which n is $x_{n,i} \in \left[A_i/b^{d_i},(A_i+1)/b^{d_i}\right)$ for all $1 \le i \le s$ satisfied. We find that $x_{n,i} \in \left[A_i/b^{d_i},(A_i+1)/b^{d_i}\right)$ means that the first d_i digits in the b-adic representation of $x_{n,i}$ are determined. In detail, let

$$\frac{A_i}{b^{d_i}} = \frac{e_1^{(i)}}{b} + \cdots + \frac{e_{d_i}^{(i)}}{b^{d_i}},$$

then

$$\frac{e_1^{(i)}}{b} + \cdots + \frac{e_{d_i}^{(i)}}{b^{d_i}} \le x_{n,i} < \frac{e_1^{(i)}}{b} + \cdots + \frac{e_{d_i}^{(i)}}{b^{d_i}} + \frac{1}{b^{d_i}},$$

that is

$$x_{n,i} = \frac{e_1^{(i)}}{b} + \cdots + \frac{e_{d_i}^{(i)}}{b^{d_i}} + \cdots .$$

Recall that by the definition of digital point sets, the jth digit of $x_{n,i}$ is given by φ^{-1} applied to the product $\mathbf{c}_j^{(i)}\mathbf{n}$ of the jth row $\mathbf{c}_j^{(i)}$ of C_i with the n-column vector $\mathbf{n} \in (\mathbb{F}_b^m)^\top$, where φ is the bijection used in the construction. Hence $\boldsymbol{x}_n \in J$ if and only if the following system of equations over $\mathbb{F}_b$ is satisfied:

$$\left.\begin{array}{rcl}
\mathbf{c}_1^{(1)}\mathbf{n} &=& \varphi(e_1^{(1)}) \\
\vdots & \vdots & \vdots \\
\mathbf{c}_{d_1}^{(1)}\mathbf{n} &=& \varphi(e_{d_1}^{(1)}) \\
\mathbf{c}_1^{(2)}\mathbf{n} &=& \varphi(e_2^{(1)}) \\
\vdots & \vdots & \vdots \\
\mathbf{c}_{d_2}^{(2)}\mathbf{n} &=& \varphi(e_{d_2}^{(1)}) \\
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
\mathbf{c}_1^{(s)}\mathbf{n} &=& \varphi(e_1^{(s)}) \\
\vdots & \vdots & \vdots \\
\mathbf{c}_{d_s}^{(s)}\mathbf{n} &=& \varphi(e_{d_s}^{(s)})
\end{array}\right\} \tag{4.8}$$

We ask, how many m-variable vectors $\mathbf{n}$ satisfy this system of $d_1+\cdots+d_s = \rho$ equations?

Since the system of row vectors $\mathbf{c}_j^{(i)}$ by definition of ρ is linearly independent, the linear system (4.8) has exactly $b^{m-\rho}$ solutions and the result is shown.

Let us now prove the strictness of the quality parameter. If $\rho = m$ then there is nothing to prove, since any $(0, m, s)$-net is strict by definition. If $\rho \leq m-1$ then by the definition of ρ there are $d_1, \ldots, d_s \in \mathbb{N}_0$ with $d_1 + \cdots + d_s = \rho + 1$ and such that

$$\mathbf{c}_1^{(1)}, \ldots, \mathbf{c}_{d_1}^{(1)}, \mathbf{c}_1^{(2)}, \ldots, \mathbf{c}_{d_2}^{(2)}, \ldots, \mathbf{c}_1^{(s)}, \ldots, \mathbf{c}_{d_s}^{(s)},$$

are linearly dependent over $\mathbb{F}_b$. But then the linear system (4.8) with $e_j^{(i)} = 0$ for all $1 \leq j \leq d_i$ and $1 \leq i \leq s$, has also $b^{m-\rho}$ solutions $\mathbf{n} \in (\mathbb{F}_b^m)^\top$ (although it consists of $\rho + 1$ equations in m variables over $\mathbb{F}_b$). This means that the elementary interval $\prod_{i=1}^s \left[0, \frac{1}{b^{d_i}}\right)$ of volume $b^{-\rho-1}$ (i.e., of order $\rho + 1$) contains $b^{m-\rho}$ points of the net and is therefore not fair. Hence the net has strict quality parameter $m - \rho$. □

Remark 4.53 According to Theorem 4.52 the strict quality parameter t of a digital net is $m - \rho$. The quantity $\rho = m - t$ is often referred to as the *strength* of a digital net.

As a consequence of Theorem 4.52 we obtain the following result.

Corollary 4.54 *Let b be a prime power. A digital net over a finite field $\mathbb{F}_b$ generated by the $m \times m$ matrices $C_1, \ldots, C_s$ is a $(0, m, s)$-net in base b if and only if for all $d_1, \ldots, d_s \in \mathbb{N}_0$ with $d_1 + \cdots + d_s = m$, the $m \times m$ matrix formed by*

the first d_1 rows of C_1 and
the first d_2 rows of C_2 and
$\vdots$
the first d_s rows of C_s

has determinant different from zero.

Therefore, the task of determining the quality parameter t is turned into determining the independence parameter ρ of the s-tuple of matrices. The advantage is now that various tools from linear algebra can be used for carrying out this task.

Example 4.55 The $(t, 4, 2)$-net over $\mathbb{Z}_2$ considered in Example 4.48 is a $(0, 4, 2)$-net over $\mathbb{Z}_2$ by Theorem 4.52.

Example 4.56 For any prime b and any $m \in \mathbb{N}$ the two $m \times m$ matrices

over $\mathbb{Z}_b$ given by

$$C_1 = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \ddots & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & \ddots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & & \iddots & 1 & 0 \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ 0 & 1 & \iddots & & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}$$

generate a digital $(0, m, 2)$-net over $\mathbb{Z}_b$. For every choice of $0 \le d \le m$ the system of vectors

$$\mathbf{c}_1^{(1)}, \dots, \mathbf{c}_d^{(1)}, \mathbf{c}_1^{(2)}, \dots, \mathbf{c}_{m-d}^{(2)},$$

where $\mathbf{c}_j^{(i)}$ denotes the jth row vector of the matrix C_i, is linearly independent over $\mathbb{Z}_b$. Hence the quality parameter $t = 0$. Indeed, the resulting digital $(0, m, 2)$-net over $\mathbb{Z}_b$ is just the two-dimensional Hammersley point set in base b.

Example 4.57 We now show that the following three $m \times m$ matrices C_1, C_2, and C_3 over $\mathbb{Z}_2$ provide, for all $m \ge 1$, a digital $(0, m, 3)$-net over $\mathbb{Z}_2$. Let

$$C_1 = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \ddots & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & \ddots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & & \iddots & 1 & 0 \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ 0 & 1 & \iddots & & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}$$

and

$$C_3 = \begin{pmatrix} \binom{0}{0} & \binom{1}{0} & \dots & \dots & \binom{m-1}{0} \\ 0 & \binom{1}{1} & \dots & \dots & \binom{m-1}{1} \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & \dots & 0 & \binom{m-2}{m-2} & \binom{m-1}{m-2} \\ 0 & \dots & \dots & 0 & \binom{m-1}{m-1} \end{pmatrix},$$

where the binomial coefficients are taken modulo 2. This example was first provided by Sobol′ in [251]. See also [170, Proof of Theorem 6.2]. We have

to show that for any choice of $d_1, d_2 \in \mathbb{N}_0$ with $d_1 + d_2 \leq m$ the vectors

$$\begin{array}{cccccccc}
(1, & 0, & \dots & \dots & \dots & \dots & \dots, & 0) \\
(0, & 1, & 0, & \dots & \dots & \dots & \dots, & 0) \\
\vdots & \ddots & \ddots & \ddots & & & & \vdots \\
(0, & \dots, & 0, & 1, & 0, & \dots & \dots, & 0) \\
 & & & \uparrow & & & & \\
 & & & d_1 & & & & \\
(0, & \dots & \dots & \dots & \dots & \dots & 0, & 1) \\
(0, & \dots & \dots & \dots & \dots & 0, & 1, & 0) \\
\vdots & & & & \iddots & \iddots & \iddots & \vdots \\
(0, & \dots & \dots & 0, & 1, & 0, & \dots, & 0) \\
 & & & & \uparrow & & & \\
 & & & & m - d_2 + 1 & & &
\end{array}$$

$$\begin{array}{cccccc}
\left(\binom{0}{0}, & \binom{1}{0}, & \dots & \dots & \dots & \dots, \ \binom{m-1}{0}\right) \\
\left(0, & \binom{1}{1}, & \binom{2}{1}, & \dots & \dots & \dots \ \binom{m-1}{1}\right) \\
\vdots & \ddots & \ddots & \ddots & & \vdots \\
\left(0, & \dots, & 0, & \binom{d_3-1}{d_3-1}, & \binom{d_3}{d_3-1}, & \dots, \ \binom{m-1}{d_3-1}\right)
\end{array}$$

are linearly independent over $\mathbb{Z}_2$ (here $d_3 := m - d_1 - d_2$). To do this, we show that the $m \times m$ matrix

$$C = \begin{pmatrix}
1 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\
0 & 1 & 0 & \dots & \dots & \dots & \dots & \dots & 0 \\
\vdots & \ddots & \ddots & \ddots & & & & & \vdots \\
0 & \dots & 0 & 1 & 0 & \dots & \dots & \dots & 0 \\
0 & \dots & \dots & \dots & \dots & \dots & \dots & 0 & 1 \\
0 & \dots & \dots & \dots & \dots & \dots & 0 & 1 & 0 \\
\vdots & & & & & \iddots & \iddots & \iddots & \vdots \\
0 & \dots & \dots & \dots & 0 & 1 & 0 & \dots & 0 \\
\binom{0}{0} & \binom{1}{0} & \dots & \dots & \dots & \dots & \dots & \binom{m-2}{0} & \binom{m-1}{0} \\
\vdots & \vdots & & & & & & \vdots & \vdots \\
\binom{0}{d_3-1} & \binom{1}{d_3-1} & \dots & \dots & \dots & \dots & \dots & \binom{m-2}{d_3-1} & \binom{m-1}{d_3-1}
\end{pmatrix},$$

where we define $\binom{a}{b} := 0$ if $b > a$, over $\mathbb{Z}_2$ has determinant 1. Developing the determinant of C along the first $d_1 + d_2$ rows yields that $|\det C| = |\det C'|$

with

$$C' = \begin{pmatrix} \binom{d_1}{0} & \cdots & \binom{m-d_2-1}{0} \\ \vdots & & \vdots \\ \binom{d_1}{d_3-1} & \cdots & \binom{m-d_2-1}{d_3-1} \end{pmatrix} \in \mathbb{Z}_2^{d_3 \times d_3}.$$

For any nonnegative integers a and k let us now consider the determinant of the matrix

$$D_{a,k} = \begin{pmatrix} \binom{a}{0} & \cdots & \binom{a+k}{0} \\ \vdots & & \vdots \\ \binom{a}{k} & \cdots & \binom{a+k}{k} \end{pmatrix} \in \mathbb{Z}_2^{(k+1)\times k+1)}.$$

For $j = k, k-1, \ldots, 1$ we successively subtract the jth column of $D_{a,k}$ from the $j+1$st column and by using the fact that $\binom{a+j}{i} - \binom{a+j-1}{i} = \binom{a+j-1}{i-1}$ we arrive at the matrix

$$D'_{a,k} = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ \binom{a}{1} & \binom{a}{0} & \binom{a+1}{0} & \ldots & \binom{a+k-1}{0} \\ \vdots & \vdots & \vdots & & \vdots \\ \binom{a}{k} & \binom{a}{k-1} & \binom{a+1}{k-1} & \ldots & \binom{a+k-1}{k-1} \end{pmatrix},$$

so that, by developing the determinant of the above matrix along the first row, we obtain $\det(D_{a,k}) = \det(D'_{a,k}) = \det(D_{a,k-1})$ and by proceeding in this way we obtain $\det(D_{a,k}) = \det(D_{a,0}) = \binom{a}{0} = 1$. The result then follows from Corollary 4.54. □

For later use we introduce a further quantity.

Definition 4.58 Let $C_1, \ldots, C_s$ be $m \times m$ matrices over $\mathbb{F}_b$. Define $\delta = \delta(C_1, \ldots, C_s)$ to be the least integer t, with $0 \le t \le m$, such that for any $d_1, \ldots, d_s \in \mathbb{N}_0$ with $d_1 + \cdots + d_s = m - t$ and any $\overline{e}_j^{(i)} \in \mathbb{F}_b$, for $1 \le j \le d_i$

and $1 \le i \le s$, the system

$$\left.\begin{array}{rcl} \mathbf{c}_1^{(1)}\mathbf{n} &=& \overline{e}_1^{(1)} \\ &\vdots& \\ \mathbf{c}_{d_1}^{(1)}\mathbf{n} &=& \overline{e}_{d_1}^{(1)} \\ \mathbf{c}_1^{(2)}\mathbf{n} &=& \overline{e}_1^{(2)} \\ &\vdots& \\ \mathbf{c}_{d_2}^{(2)}\mathbf{n} &=& \overline{e}_{d_2}^{(2)} \\ &\vdots& \\ \mathbf{c}_1^{(s)}\mathbf{n} &=& \overline{e}_1^{(s)} \\ &\vdots& \\ \mathbf{c}_{d_s}^{(s)}\mathbf{n} &=& \overline{e}_{d_s}^{(s)} \end{array}\right\} \tag{4.9}$$

where $\mathbf{c}_j^{(i)}$ denotes the jth row vector of C_i, has exactly b^t solutions $\mathbf{n} \in (\mathbb{F}_b^m)^\top$.

Obviously in the above definition it suffices to consider the homogeneous system only, i.e., $\overline{e}_j^{(i)} = 0$ for all i and j. This system has exactly b^t solutions, if and only if each system in (4.9) has exactly b^t solutions. This follows from elementary properties of systems of linear equations.

The proof of the following result is left as an exercise (see Exercise 4.6).

Lemma 4.59 *Let b be a prime power. Any $m \times m$ matrices $C_1, \ldots, C_s$ over $\mathbb{F}_b$ generate a strict digital (δ, m, s)-net over $\mathbb{F}_b$.*

Propagation rules for digital nets

We have seen several propagation rules for (t, m, s)-nets in Section 4.2. We show now that certain propagation rules also hold for digital nets. For instance:

1. Any digital (t, m, s)-net over $\mathbb{F}_b$ is a digital (t', m, s)-net over $\mathbb{F}_b$ for all $t' \ge t$.
2. If the matrices $C_1, \ldots, C_s$ generate a digital (t, m, s)-net over $\mathbb{F}_b$ and if we take any $s' \le s$ of these matrices, then these matrices form a digital (t, m, s')-net over $\mathbb{F}_b$.

Again it is more subtle to provide suitable propagation rules for digital (t, m, s)-nets concerning the parameter m. The following propagation rule was first given (for arbitrary bases b) in [234, Lemma 3].

Theorem 4.60 *Let b be a prime power. If there exists a digital (t, m, s)-net over $\mathbb{F}_b$, then for each n with $t \le n \le m$, there exists a digital (t, n, s)-net over $\mathbb{F}_b$.*

For the proof of this result we need the following lemmas.

Lemma 4.61 *Let b be a prime power and let a (strict) digital (t, m, s)-net over $\mathbb{F}_b$ be generated by the $m \times m$ matrices $C_1, \ldots, C_s$. Let Z be a nonsingular $m \times m$ matrix over $\mathbb{F}_b$. Then the matrices $C_1', \ldots, C_s'$ with $C_i' := C_i Z$ also generate a (strict) digital (t, m, s)-net over $\mathbb{F}_b$. Indeed, they generate the same digital net, only with order of points changed.*

The proof of this result is left as an exercise (see Exercise 4.7).

Lemma 4.62 *Let b be a prime power. If there exists a digital (t, m, s)-net over $\mathbb{F}_b$, then for any given nonsingular $m \times m$ matrix Y_s over $\mathbb{F}_b$, there are nonsingular $m \times m$ matrices $D_1, \ldots, D_s$ over $\mathbb{F}_b$ with $D_s = Y_s$, generating a digital (t, m, s)-net over $\mathbb{F}_b$.*

Proof Let $C_1, \ldots, C_s$ be an s-tuple of $m \times m$ matrices generating a digital (t, m, s)-net over $\mathbb{F}_b$. Theorem 4.52 implies that the linear independence parameter ρ of $C_1, \ldots, C_s$ satisfies $\rho = m - t$, and hence for each C_i the first $m-t$ rows are linearly independent. We now generate new $m \times m$ matrices $\widetilde{C}_i$ by removing the last t rows of C_i and by completing the remaining $m-t$ rows by t arbitrary rows, such that all m rows of the new matrix $\widetilde{C}_i$ are linearly independent. This is possible since $\mathbb{F}_b$ is a field. The matrices $\widetilde{C}_1, \ldots, \widetilde{C}_s$ again generate a (t, m, s)-net over $\mathbb{F}_b$. Since $\widetilde{C}_s$ is invertible, there exists a nonsingular $m \times m$ matrix Z over $\mathbb{F}_b$ such that $\widetilde{C}_s Z = Y_s$. Let the $m \times m$ matrices $D_1, \ldots, D_s$ be defined by $D_i = \widetilde{C}_i Z$, i.e., in particular $D_s = Y_s$. By Lemma 4.61 the matrices $D_1, \ldots, D_s$ again generate a digital (t, m, s)-net over $\mathbb{F}_b$. □

Proof of Theorem 4.60 By Lemma 4.62 we may assume that the given digital (t, m, s)-net over $\mathbb{F}_b$ is generated by the nonsingular $m \times m$ matrices $C_1, \ldots, C_s$ over $\mathbb{F}_b$, where

$$C_s = E_m' := \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & & \iddots & 1 & 0 \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ 0 & 1 & \iddots & & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{pmatrix} \in \mathbb{F}_b^{m \times m}.$$

Define now $n \times n$ matrices $D_1, \ldots, D_s$ over $\mathbb{F}_b$ by setting $D_i := C_i^{(n)}$ (i.e.,

the left upper $n \times n$ sub-matrix of C_i) for $1 \le i \le s-1$ and $D_s := E'_n$. We show that $D_1, \ldots, D_s$ generate a digital (t, n, s)-net over $\mathbb{F}_b$.

Let $d_1, \ldots, d_s \in \mathbb{N}_0$ with $d_1 + \cdots + d_s = n - t$ be arbitrarily given. Take the system of the first d_1 rows of D_1, the first d_2 rows of D_2, ..., the first d_s rows of D_s. For simplicity we set $d_1 + \cdots + d_{s-1} =: d$ and denote the first d vectors above by $\mathbf{a}_j^{(n)} \in \mathbb{F}_b^n$ for $1 \le j \le d$. They are the projection of the corresponding m-dimensional vectors $\mathbf{a}_j$ from the matrices C_i from above. We write $\mathbf{a}_j = (a_{j,1}, \ldots, a_{j,n}, |a_{j,n+1}, \ldots, a_{j,m}) = (\mathbf{a}_j^{(n)} | \widetilde{\mathbf{a}}_j^{(m-n)})$. Since $C_1, \ldots, C_{s-1}$ and $C_s = E'_m$ generate a (t, m, s)-net over $\mathbb{F}_b$, the system of $m - t$ vectors from $\mathbb{F}_b^m$ given by

$$
\begin{array}{cccccc|cccc}
(a_{1,1}, & \ldots, & \ldots, & \ldots, & \ldots, & a_{1,n}, & a_{1,n+1}, & \ldots, & \ldots, & a_{1,m}), \\
\vdots & & & & & \vdots & \vdots & & & \vdots \\
(a_{d,1}, & \ldots, & \ldots, & \ldots, & \ldots, & a_{d,n}, & a_{d,n+1}, & \ldots, & \ldots, & a_{d,m}), \\
(0, & \ldots, & \ldots, & \ldots, & \ldots, & 0, & 0, & \ldots, & 0, & 1), \\
(0, & \ldots, & \ldots, & \ldots, & \ldots, & \vdots, & \vdots & \iddots & 1, & 0), \\
\vdots & & & & & \vdots & 0, & \iddots & \iddots & \vdots \\
(0, & \ldots, & \ldots, & \ldots, & \ldots, & 0, & 1, & 0, & \ldots, & 0), \\
(0, & \ldots, & \ldots, & \ldots & 0, & 1, & 0, & \ldots, & \ldots, & 0), \\
\vdots & & & \iddots & \iddots & \iddots & \vdots & & & \vdots \\
(0, & \ldots, & 0, & 1, & 0, & \ldots, & 0, & \ldots, & \ldots, & 0),
\end{array}
$$

where the "1" in the last vector is the $t + d + 1$st component, is linearly independent over $\mathbb{F}_b$ (note that $d \le n - t$ and therefore $t + d \le n$). But then, as it is obvious by the above scheme, $\mathbf{a}_1^{(n)}, \ldots, \mathbf{a}_d^{(n)}$ and the first $n - t - d$ rows of E'_n must be linearly independent over $\mathbb{F}_b$ and the result follows. □

More detailed propagation rules for digital nets are presented in Chapter 9.

Structural results for digital nets

We give in the following some general structural results for digital nets. We have seen in the example shown in Figure 4.11 in Section 4.2 that the addition modulo 1 of a fixed s-dimensional point $\boldsymbol{x}$ to all the points of a (t, m, s)-net in base b, although it does not disturb the net property, it does change in general the (strict) quality parameter t. The principal *digital net* property, however, by shifting the net in general is destroyed. This, for example, can be seen by the fact that any coordinate of any point of a digital (t, m, s)-net in base b is of the form a/b^m, where a is an integer.

Hence, for example, addition of a vector whose coordinates are not all of this form destroys the digital net property. Another reason is that any digital net contains the origin. Hence shifting the net in a way which removes the origin from the point set destroys the digital net property. Any elementary interval in base b is half-open at the right-upper boundary. The right-upper boundary of an elementary interval in base b of order less than or equal to m (i.e., of volume larger or equal b^{-m}) in all coordinates is of the form a/b^m. Therefore any element of a digital (t, m, s)-net in base b has distance from the right-upper boundary, of any elementary interval of order less than or equal to m, of at least b^{-m}. From this fact the following stability result for digital (t, m, s)-nets in a base b follows.

Lemma 4.63 *Let b be a prime power and let $\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{b^m-1}\}$ with $\boldsymbol{x}_n := (x_{n,1}, \ldots, x_{n,s})$ be a strict digital (t, m, s)-net over $\mathbb{F}_b$. Let $\varepsilon_{n,i}$ for $1 \le i \le s$ and $0 \le n \le b^m - 1$ be nonnegative reals with $\varepsilon_{n,i} < b^{-m}$ for all n and i. Then $\{\boldsymbol{y}_0, \ldots, \boldsymbol{y}_{b^m-1}\}$ with $\boldsymbol{y}_n := (x_{n,1} + \varepsilon_{n,1}, \ldots, x_{n,s} + \varepsilon_{n,s})$ is a strict (t, m, s)-net in base b.*

Remark 4.64 Indeed this property holds for all (t, m, s)-nets in base b whose points have coordinates of the form a/b^m with integers $0 \le a < b^m$.

Another form of shifting a digital net is of higher relevance. Recall the scheme for generating a digital (t, m, s)-net over $\mathbb{F}_b$. Let the integer n be such that $0 \le n \le b^m - 1$. Then

$$n \to \mathbf{n} \in (\mathbb{F}_b^m)^\top \to C_i \mathbf{n} = \begin{pmatrix} \overline{y}_{n,i,1} \\ \vdots \\ \overline{y}_{n,i,m} \end{pmatrix} \in (\mathbb{F}_b^m)^\top \to x_{n,i} \in [0, 1).$$

Instead of shifting $x_{n,i}$ like above, let us now shift the column vector

$$\begin{pmatrix} \overline{y}_{n,i,1} \\ \vdots \\ \overline{y}_{n,i,m} \end{pmatrix} \in (\mathbb{F}_b^m)^\top$$

by a fixed column vector over $\mathbb{F}_b$, say

$$\begin{pmatrix} \overline{\sigma}_{i,1} \\ \vdots \\ \overline{\sigma}_{i,m} \end{pmatrix} \in (\mathbb{F}_b^m)^\top.$$

That is, instead of $x_{n,i}$ consider $z_{n,i}$, which is obtained by

$$\begin{pmatrix} \overline{z}_{n,i,1} \\ \vdots \\ \overline{z}_{n,i,m} \end{pmatrix} = \begin{pmatrix} \overline{y}_{n,i,1} + \overline{\sigma}_{i,1} \\ \vdots \\ \overline{y}_{n,i,m} + \overline{\sigma}_{i,m} \end{pmatrix}$$

and

$$z_{n,i} = \frac{\varphi^{-1}(\overline{z}_{n,i,1})}{b} + \cdots + \frac{\varphi^{-1}(\overline{z}_{n,i,m})}{b^m}.$$

We introduce a slightly more general concept here.

Definition 4.65 Let b be a prime power and $\varphi : \{0, \ldots, b-1\} \to \mathbb{F}_b$ be a bijection with $\varphi(0) = \overline{0}$. For $x = \sum_{i=1}^{\infty} \frac{\xi_i}{b^i} \in [0,1)$ and $\sigma = \sum_{i=1}^{\infty} \frac{\varsigma_i}{b^i} \in [0,1)$, where $\xi_i, \varsigma_i \in \{0, \ldots, b-1\}$, we define the ($b$-adic) *digital shifted point* y by $y = x \oplus_{b,\varphi} \sigma := \sum_{i=1}^{\infty} \frac{\eta_i}{b^i}$ where $\eta_i = \varphi^{-1}(\varphi(\xi_i) + \varphi(\varsigma_i))$, and where the "+" is addition in $\mathbb{F}_b$.

For higher dimensions $s > 1$ let $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_s) \in [0,1)^s$. For $\boldsymbol{x} = (x_1, \ldots, x_s) \in [0,1)^s$ we define the (b-adic) *digital shifted point* $\boldsymbol{y}$ by $\boldsymbol{y} = \boldsymbol{x} \oplus_{b,\varphi} \boldsymbol{\sigma} = (x_1 \oplus_{b,\varphi} \sigma_1, \ldots, x_s \oplus_{b,\varphi} \sigma_s)$.

In the following b and the bijection φ are considered to be fixed and therefore we simply write $\oplus$ instead of $\oplus_{b,\varphi}$.

Definition 4.66 Let b be a prime power and $\varphi : \{0, \ldots, b-1\} \to \mathbb{F}_b$ be a bijection with $\varphi(0) = \overline{0}$. For a point set $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$ in $[0,1)^s$ and a $\boldsymbol{\sigma} \in [0,1)^s$ the point set $\mathcal{P}_{\boldsymbol{\sigma}} = \{\boldsymbol{x}_0 \oplus \boldsymbol{\sigma}, \ldots, \boldsymbol{x}_{N-1} \oplus \boldsymbol{\sigma}\}$ is called the (b-adic) *digitally shifted point set* $\mathcal{P}$, or the (b-adic) *digitally shifted version of* $\mathcal{P}$. The vector $\boldsymbol{\sigma} \in [0,1)^s$ is called a (b-adic) *digital shift*.

If we use a digital shift in conjunction with a (t,m,s)-net, then they are always assumed to be in the same base b. Therefore, if it is clear with respect to which base b a point is shifted, we may omit the phrase "b-adic".

We show now that a digital shift preserves the (t,m,s)-net structure.

Lemma 4.67 *Let b be a prime power, $\varphi : \{0, \ldots, b-1\} \to \mathbb{F}_b$ a bijection with $\varphi(0) = \overline{0}$ and let $\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{b^m-1}\}$ be a (strict) (t,m,s)-net in base b, $\boldsymbol{x}_n = (x_{n,1}, \ldots, x_{n,s})$ for $0 \le n < b^m$, and let $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_s) \in [0,1)^s$. Then the digitally shifted point set formed by the points $\boldsymbol{y}_n = \boldsymbol{x}_n \oplus \boldsymbol{\sigma}$, $0 \le n < b^m$, is again a (strict) (t,m,s)-net in base b with probability one with respect to the Lebesgue measure of $\boldsymbol{\sigma}$'s. (If the σ_i's have only finitely many b-adic digits different from zero, then the assertion is always true.)*

Proof First we note that for any $x \in [0,1)$ the set of all $\sigma \in [0,1)$, for which

the b-adic expansion of $x \oplus \sigma$ has only finitely many digits different from $b-1$, is countable. In fact, if ξ_j denotes the digits in the b-adic expansion of x and ς_j denotes the digits in the b-adic expansion of σ, then $x \oplus \sigma$ has only finitely many digits different from $b-1$ iff there is an index j_0 such that for all $j \geq j_0$ we have $\varphi(\xi_j) + \varphi(\varsigma_j) = \varphi(b-1) \in \mathbb{F}_b$ and this holds if and only if $\varsigma_j = \varphi^{-1}(\varphi(b-1) - \varphi(\xi_j)) \in \mathbb{F}_b$ for all $j \geq j_0$. Thus the Lebesgue measure of this set is zero and the probability that this case occurs is zero as well.

For $1 \leq i \leq s$ let $\sigma_i = \frac{\varsigma_{i,1}}{b} + \frac{\sigma_{i,2}}{b^2} + \cdots$. Further, for $0 \leq n < b^m$ and $1 \leq i \leq s$ let $x_{n,i} = \frac{\xi_{n,i,1}}{b} + \frac{\xi_{n,i,2}}{b^2} + \cdots$ and $y_{n,i} = \frac{\eta_{n,i,1}}{b} + \frac{\eta_{n,i,2}}{b^2} + \cdots$, where for $k \geq 1$,

$$\eta_{n,i,k} = \varphi^{-1}(\varphi(\xi_{n,i,k}) + \varphi(\varsigma_{i,k})).$$

In the following we assume that infinitely many of the $\eta_{n,i,1}, \eta_{n,i,2}, \ldots$ are different from $b-1$. As shown above this occurs with probability one. Let

$$J = \prod_{i=1}^{s} \left[\frac{A_i}{b^{d_i}}, \frac{A_i+1}{b^{d_i}} \right)$$

be an elementary interval of volume b^{t-m}, i.e., $d_1, \ldots, d_s \in \mathbb{N}_0$ with $d_1 + \cdots + d_s = m - t$ and integers $A_1, \ldots, A_s$ with $0 \leq A_i < b^{d_i}$ for $1 \leq i \leq s$, and let

$$\frac{A_i}{b^{d_i}} = \frac{A_{i,1}}{b} + \cdots + \frac{A_{i,d_i}}{b^{d_i}}.$$

Then the point $\boldsymbol{y}_n$ is contained in J if and only if

$$\eta_{n,i,k} = A_{i,k} \quad \text{for all } 1 \leq k \leq d_i \text{ and } 1 \leq i \leq s,$$

and this is true if and only if

$$\xi_{n,i,k} = \varphi^{-1}(\varphi(A_{i,k}) - \varphi(\varsigma_{i,k})) \quad \text{for all } 1 \leq k \leq d_i \text{ and } 1 \leq i \leq s. \tag{4.10}$$

Let now $\overline{B}_{i,k} \in \mathbb{F}_b$ such that

$$\overline{B}_{i,k} = \varphi(A_{i,k}) - \varphi(\varsigma_{i,k}),$$

and let

$$\frac{B_i}{b^{d_i}} = \frac{B_{i,1}}{b} + \cdots + \frac{B_{i,d_i}}{b^{d_i}},$$

where $B_{i,k} = \varphi^{-1}(\overline{B}_{i,k})$ for $1 \leq k \leq d_i$ and $1 \leq i \leq s$. Then (4.10) is equivalent to

$$\boldsymbol{x}_n \in M := \prod_{i=1}^{s} \left[\frac{B_i}{b^{d_i}}, \frac{B_i+1}{b^{d_i}} \right).$$

Now M is again an elementary interval of volume b^{t-m}. Since $\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{b^m-1}\}$ forms a (t, m, s)-net in base b, it follows that M contains exactly b^t points of $\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{b^m-1}\}$. Therefore J contains exactly b^t points of $\{\boldsymbol{y}_0, \ldots, \boldsymbol{y}_{b^m-1}\}$ and hence this point set is a (t, m, s)-net in base b.

If $t = 0$, then the $\boldsymbol{y}_n$, $0 \le n < b^m$, form a strict net. If $t \ge 1$, then let now

$$M := \prod_{i=1}^{s} \left[\frac{B_i}{b^{d_i}}, \frac{B_i + 1}{b^{d_i}} \right)$$

be an elementary interval of order $m - t + 1$, such that M does not contain exactly b^{t-1} of the elements of the strict (t, m, s)-net $\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{b^m-1}\}$. Let $B_i = \frac{B_{i,1}}{b} + \cdots + \frac{B_{i,d_i}}{b^{d_i}}$. Defining now in the opposite way $A_i = \frac{A_{i,1}}{b} + \cdots + \frac{A_{i,d_i}}{b^{d_i}}$ such that $\varphi(A_{i,k}) = \varphi(B_{i,k}) + \varphi(\varsigma_{i,k}) \in \mathbb{F}_b$ for $1 \le k \le d_i$ and $1 \le i \le s$, then as above

$$\boldsymbol{x}_n \in M \quad \text{if and only if} \quad \boldsymbol{x}_n \in J := \textstyle\prod_{i=1}^{s} \left[\frac{A_i}{b^{d_i}}, \frac{A_i+1}{b^{d_i}} \right).$$

Therefore the strictness of the net $\{\boldsymbol{y}_0, \ldots, \boldsymbol{y}_{b^m-1}\}$ follows. □

Remark 4.68 Note that for a given net the digital shift $\boldsymbol{\sigma} \in [0, 1)^s$ can be chosen such that the origin is not contained in the shifted version of the net any more. Hence, in general, the digitally shifted version of a digital net is not a digital net.

There are several variants of digital shifts. We introduce the so-called digital shift of depth m and a simplified digital shift for digital nets. Such shifts are used later in Chapter 16 when we show the existence of digital nets which achieve the best possible order of the L_2-discrepancy.

Definition 4.69 Let b be a prime power and let $\varphi : \{0, \ldots, b-1\} \to \mathbb{F}_b$ be a bijection with $\varphi(0) = \overline{0}$. Let $\mathcal{P}_{b^m} = \{x_0, \ldots, x_{b^m-1}\}$ be a digital $(t, m, 1)$-net over $\mathbb{F}_b$ and let

$$x_n = \frac{x_{n,1}}{b} + \frac{x_{n,2}}{b^2} + \cdots + \frac{x_{n,m}}{b^m}$$

be the b-adic digit expansion of x_n. Choose $\sigma = \frac{\varsigma_1}{b} + \cdots + \frac{\varsigma_m}{b^m}$ with $\varsigma_i \in \mathbb{F}_b$ and define

$$z_{n,i} := \varphi^{-1}(\varphi(x_{n,i}) + \varphi(\varsigma_i)) \qquad \text{for } 1 \le i \le m.$$

Further, for $0 \le n < b^m$, choose $\delta_n \in [0, b^{-m})$. Then the digitally shifted point set $\widetilde{\mathcal{P}}_{b^m} = \{z_0, \ldots, z_{b^m-1}\}$ is defined by

$$z_n = \frac{z_{n,1}}{b} + \cdots + \frac{z_{n,m}}{b^m} + \delta_n.$$

Such a digital shift is called a *digital shift of depth* m.

For higher dimensions $s > 1$ each coordinate is shifted independently by a digital shift of depth m.

This means that one applies the same digital shift to the first m digits, whereas the following digits are shifted independently for each x_n. In other words, a digital shift of depth m is a combination of a digital shift $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_s)$ where the σ_i's are of the form $\sigma_i = \varsigma_{i,1}/b + \cdots + \varsigma_{i,m}/b^m$ with $\varsigma_{i,j} \in \mathbb{F}_b$ for $1 \le j \le m$ and $1 \le i \le s$, and a geometric shift as used in Lemma 4.63.

We also introduce a simplified version of a digital shift (of depth m).

Definition 4.70 With the notation from Definition 4.69 above, we define a digitally shifted point set $\widehat{\mathcal{P}}_{b^m} = \{z_0, \ldots, z_{b^m-1}\}$ by

$$z_n = \frac{z_{n,1}}{b} + \cdots + \frac{z_{n,m}}{b^m} + \frac{1}{2b^m}.$$

This means we apply the same digital shift to the first m digits and then we add to each point the quantity $1/(2b^m)$. Such a shift is called a *simplified digital shift (of depth m)*. For higher dimensions $s > 1$ each coordinate is shifted independently by a simplified digital shift.

Geometrically, the simplified digital shift of depth m means that the shifted points are no longer on the left boundary of elementary intervals of the form $\prod_{i=1}^{s}[A_i/b^m, (A_i+1)/b^m)$, but they are moved to the midpoints of such intervals. Note that for the simplified digital shift we only have b^m possibilities, which means only sm digits need to be selected in performing a simplified digital shift. In comparison, the digital shift of depth m requires infinitely many digits.

It can be shown that a (strict) digital (t, m, s)-net over $\mathbb{F}_b$, which is shifted by a digital shift of depth m or a simplified digital shift independently in each coordinate is again a (strict) (t, m, s)-net in base b with the same quality parameter t (with probability one in the case of a digital shift of depth m). See Exercise 4.9 and Exercise 4.10.

Example 4.71 Consider the eight elements of the digital $(0, 3, 2)$-net over $\mathbb{Z}_2$ shown on the left-hand side of Figure 4.22, which are generated by

$$C_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad C_2 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Applying a 2-adic digital shift $\boldsymbol{\sigma} = (\sigma_1, \sigma_2)$ with $\sigma_1 = 1/2$ and $\sigma_2 = 7/8$

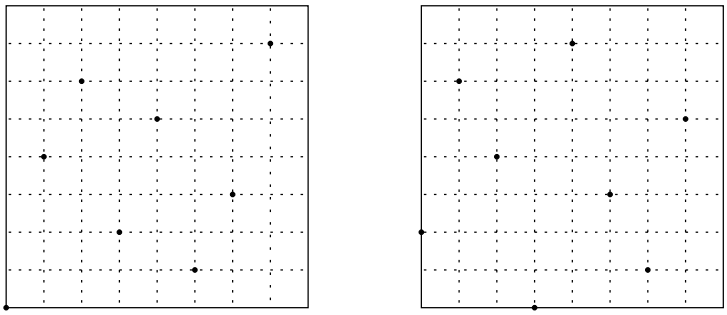

Figure 4.22 A digital $(0,3,2)$-net over $\mathbb{Z}_2$ and its digitally shifted version.

then gives, for example, $\boldsymbol{y}_3 = (y_{3,1}, y_{3,2})$, where

$$y_{3,1} \leftarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

and

$$y_{3,2} \leftarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

and hence $\boldsymbol{y}_3 = (7/8, 5/8)$.

We obtain the point set shown on the right-hand side of Figure 4.22, which obviously cannot be obtained by the original digital net by an ordinary translation, and which is not a digital net any more (as it does not contain the origin).

Now let us disturb this point set in the "positive direction" by individual quantities $\delta_{n,i}$ less than $1/8$ in each coordinate i for $0 \le n < 8$ (that is, we have a digital shift of depth 3). Then we arrive, for example, at the point set shown in Figure 4.23, which is still a strict $(0,3,2)$-net in base 2.

A further very important structural property of digital nets is their group structure which was first used by Larcher, Niederreiter & Schmid [139].

Let b be a prime power and let $\varphi : \{0, \ldots, b-1\} \to \mathbb{F}_b$ be a bijection with $\varphi(0) = \overline{0}$. The s-dimensional unit cube is an abelian group with respect to the digit-wise b-adic addition $\oplus$ as used in Definition 4.65. For $x, y \in [0,1)$ let $x = \frac{\xi_1}{b} + \frac{\xi_2}{b^2} + \cdots$ and $y = \frac{\eta_1}{b} + \frac{\eta_2}{b^2} + \cdots$ be their b-adic expansions (with $\xi_i \ne b-1$ for infinitely many i and $\eta_j \ne b-1$ for infinitely many j). Then $x \oplus_{b,\varphi} y := \frac{\zeta_1}{b} + \frac{\zeta_2}{b^2} + \cdots$ with

$$\zeta_j = \varphi^{-1}(\varphi(\xi_j) + \varphi(\eta_j)) \quad \text{for} \quad j \in \mathbb{N}.$$

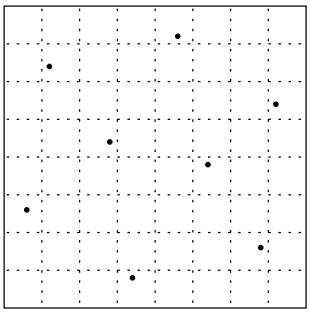

Figure 4.23 A digital shift of depth 3 applied to a digital $(0, 3, 2)$-net in base 2.

For vectors $\boldsymbol{x}, \boldsymbol{y} \in [0,1)^s$ the b-adic addition $\boldsymbol{x} \oplus_{b,\varphi} \boldsymbol{y}$ is defined component wise.

As before, the base b and the bijection φ are considered to be fixed and therefore we simply write $\oplus$ instead of $\oplus_{b,\varphi}$. If we use the b-adic addition $\oplus = \oplus_{b,\varphi}$ in conjunction with a digital net, then we always assume that b is the base of the digital net and φ is the bijection from the construction of the digital net.

Now we consider the natural continuation of $\oplus$ to $[0,1)^s$ which we denote again by $\oplus$. Obviously $([0,1)^s, \oplus)$ is an abelian group. We then have the following lemma.

Lemma 4.72 *Let b be a prime power and let $\varphi : \{0, \ldots, b-1\} \to \mathbb{F}_b$ be a bijection with $\varphi(0) = \overline{0}$. Any digital (t, m, s)-net over $\mathbb{F}_b$ is a subgroup of $([0,1)^s, \oplus)$. If the points of the digital net are pairwise different, then this subgroup is isomorphic to $\mathbb{F}_b^m$.*

Proof Any column vector

$$\mathbf{n} = \begin{pmatrix} \overline{n}_0 \\ \vdots \\ \overline{n}_{m-1} \end{pmatrix} \in (\mathbb{F}_b^m)^\top$$

uniquely represents an integer $n := n_0 + n_1 b + \cdots + n_{m-1} b^{m-1}$ from $\{0, \ldots, b^m - 1\}$ via $n_i = \varphi^{-1}(\overline{n}_i)$ for $0 \le i < m$, and to any such integer belongs a net element $\boldsymbol{x}_n$.

We show that the mapping

$$\Psi : (\mathbb{F}_b^m)^\top \to \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{b^m-1}\},\ \mathbf{n} \mapsto \boldsymbol{x}_n$$

is a group-isomorphism from the additive group of $\mathbb{F}_b^m$ to $(\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{b^m-1}\}, \oplus)$.

Let

$$\mathbf{n} := \begin{pmatrix} \overline{n}_0 \\ \vdots \\ \overline{n}_{m-1} \end{pmatrix} \quad \text{and} \quad \mathbf{l} := \begin{pmatrix} \overline{l}_0 \\ \vdots \\ \overline{l}_{m-1} \end{pmatrix}$$

be two elements from $(\mathbb{F}_b^m)^\top$. Then the property $\Psi(\mathbf{n}+\mathbf{l}) = \Psi(\mathbf{n}) \oplus \Psi(\mathbf{l})$ easily follows from the fact that for any $m \times m$ matrix C over $\mathbb{F}_b$ we have $C(\mathbf{n}+\mathbf{l}) = C\mathbf{n} + C\mathbf{l}$. If the points of the digital net are pairwise different, then the mapping Ψ is surjective and therefore also injective, because $|\mathbb{F}_b^m| = |\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{b^m-1}\}|$. The result follows. □

Remark 4.73 With the notation of the b-adic addition we may interpret the digitally b-adic digital shifting of a digital net as a translation of the digital net with respect to $\oplus$ along a certain translation vector of $[0,1)^s$. From the group structure of the digital net it follows that a digital net remains unchanged by translation with respect to $\oplus$ if and only if the translation vector is an element of the digital net.

Example 4.74 In the shifting example above the translation vector was given by $(1/2, 7/8)$ in $[0,1)^2$. This point does not belong to the original digital net, so that the digitally shifted net is different from the original one.

For the following let b be a prime number and identify the finite field $\mathbb{F}_b$ with $\mathbb{Z}_b$. In this case we show how b-adic Walsh functions are linked to digital nets over $\mathbb{Z}_b$. This connection is very important for the analysis of the discrepancy of digital nets and of the worst-case error of QMC rules using digital nets in certain function spaces.

Let $\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{b^m-1}\}$ be a digital (t,m,s)-net over $\mathbb{Z}_b$. By Corollary A.7, for all $\boldsymbol{k} \in \mathbb{N}_0^s$, we have

$${}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_h \oplus_b \boldsymbol{x}_i) = {}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_h)\, {}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_i)$$

and hence ${}_b\mathrm{wal}_{\boldsymbol{k}}$ is a character on the group $\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{b^m-1}\}$. Now we can prove the following very important character property of Walsh functions.

Lemma 4.75 *Let b be a prime and let $\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{b^m-1}\}$ be a digital (t,m,s)-net over $\mathbb{Z}_b$ generated by the $m \times m$ matrices $C_1, \ldots, C_s$ over $\mathbb{Z}_b$. Then for a $\boldsymbol{k} = (k_1, \ldots, k_s) \in \{0, \ldots, b^m-1\}^s$ we have*

$$\sum_{h=0}^{b^m-1} {}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_h) = \begin{cases} b^m & \text{if } C_1^\top \mathbf{k}_1 + \cdots + C_s^\top \mathbf{k}_s = \mathbf{0}, \\ 0 & \text{otherwise}, \end{cases}$$

where for $k \in \{0, \ldots, b^m-1\}$ we denote by $\mathbf{k}$ the m-dimensional column

vector of b-adic digits of k, i.e., $\mathbf{k} \in (\mathbb{Z}_b^m)^\top$, *and* $\mathbf{0}$ *denotes the zero-vector in* $(\mathbb{Z}_b^m)^\top$.

Proof Since ${}_b\mathrm{wal}_{\boldsymbol{k}}$ is a character, we obtain, by using Lemma 4.72, that

$$\sum_{h=0}^{b^m-1} {}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_h) = \begin{cases} b^m & \text{if } {}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_h) = 1 \text{ for all } 0 \le h < b^m, \\ 0 & \text{otherwise.} \end{cases}$$

We have ${}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_h) = 1$ for all $0 \le h < b^m$ if and only if

$$\sum_{i=1}^{s} \mathbf{k}_i \cdot \mathbf{x}_{h,i} = 0 \text{ for all } 0 \le h < b^m,$$

where $\mathbf{k}_i$ is the m-dimensional column vector of b-adic digits of k_i and $\mathbf{x}_{h,i}$ denotes the m-dimensional column vector of b-adic digits of the ith component of $\boldsymbol{x}_h$. From the construction of the digital net we find that $\mathbf{x}_{h,i} = C_i\mathbf{h}$ and hence ${}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_h) = 1$ for all $0 \le h < b^m$ if and only if

$$\sum_{i=1}^{s} \mathbf{k}_i \cdot C_i\mathbf{h} = 0 \text{ for all } 0 \le h < b^m,$$

where $\mathbf{h}$ denotes the row vector of b-adic digits of h. This is satisfied if and only if

$$C_1^\top \mathbf{k}_1 + \cdots + C_s^\top \mathbf{k}_s = \mathbf{0}. \qquad \square$$

A generalisation of Lemma 4.75 to the case of digital nets over $\mathbb{F}_b$ with prime power b can be found in [219, Lemma 2.5]. In this case one requires the more general concept of Walsh functions over the finite field $\mathbb{F}_b$.

Following from Lemma 4.75, we introduce the notion of a so-called dual net, which is, in this form, due to Niederreiter and Pirsic [187].

Definition 4.76 Let b be a prime. For a digital net with generating matrices $C_1, \ldots, C_s$ over $\mathbb{Z}_b$ we call the matrix $C = (C_1^\top | \ldots | C_s^\top) \in \mathbb{Z}_b^{m \times sm}$ the *overall generating matrix* of the digital net. The corresponding *dual net* is defined by

$$\mathcal{D} = \mathcal{D}(C_1, \ldots, C_s) := \{\boldsymbol{k} \in \{0, \ldots, b^m - 1\}^s \,:\, C_1^\top \mathbf{k}_1 + \cdots + C_s^\top \mathbf{k}_s = \mathbf{0}\},$$

where $\boldsymbol{k} = (k_1, \ldots, k_s)$ and for $1 \le i \le s$ we denote by $\mathbf{k}_i$ the m-dimensional column vector of b-adic digits of $k_i \in \{0, \ldots, b^m - 1\}$. Furthermore, let

$$\mathcal{D}' = \mathcal{D}'(C_1, \ldots, C_s) := \mathcal{D} \setminus \{\mathbf{0}\}.$$

Remark 4.77 Sometimes we also use the definition

$$\mathcal{D}_\infty = \mathcal{D}_\infty(C_1, \ldots, C_s) := \{\boldsymbol{k} \in \mathbb{N}_0^s \,:\, \mathrm{tr}_m(\boldsymbol{k}) \in \mathcal{D}(C_1, \ldots, C_s)\}$$

$$= \{\boldsymbol{k} \in \mathbb{N}_0^s \,:\, C_1^\top \mathrm{tr}_m(\mathbf{k}_1) + \cdots + C_s^\top \mathrm{tr}_m(\mathbf{k}_s) = \mathbf{0}\},$$

where for $k \in \mathbb{N}_0$ with b-adic expansion $k = \sum_{j\geq 0} \kappa_j b^j$ we write $\mathrm{tr}_m(k) = \kappa_0 + \kappa_1 b + \cdots + \kappa_{m-1} b^{m-1}$ and $\mathrm{tr}_m(\mathbf{k}) := (\kappa_0, \ldots, \kappa_{m-1})^\top \in (\mathbb{Z}_b^m)^\top$ and for $\boldsymbol{k} = (k_1, \ldots, k_s)$ we write $\mathrm{tr}_m(\boldsymbol{k}) = (\mathrm{tr}_m(k_1), \ldots, \mathrm{tr}_m(k_s)) \in \{0, \ldots, b^m - 1\}^s$. Again we speak of the dual net.

Duality for digital nets was first introduced and studied by Niederreiter & Pirsic [187] and, in a more specialised setting, by Skriganov [241], see also [22]. Our definition of a dual net here corresponds to the definitions given in [187]. We are concerned with duality theory for digital nets in more detail in Chapter 7.

Digital (t, s)*- and* $(\mathbf{T}, s)$*-sequences*

To construct a digital $(\mathbf{T}, s)$-sequence in prime power base b, we again use a finite field $\mathbb{F}_b$ and a bijection $\varphi : \{0, \ldots, b-1\} \to \mathbb{F}_b$ with $\varphi(0) = \overline{0}$, and we speak then of a "digital $(\mathbf{T}, s)$-sequence over $\mathbb{F}_b$". (Again, for arbitrary b one has to choose a finite commutative Ring R with identity of order b, see [135, 139, 175] for more information.) If b is a prime, we identify $\mathbb{F}_b$ with $\mathbb{Z}_b$ and we omit the bijection φ and the bar.

Let now b be a prime power. To generate a digital $(\mathbf{T}, s)$-sequence over $\mathbb{F}_b$ we first have to choose $\mathbb{N} \times \mathbb{N}$ matrices $C_1, \ldots, C_s$ (one for each component) over $\mathbb{F}_b$. That is, matrices of the form

$$C = \begin{pmatrix} c_{11} & c_{12} & c_{13} & \ldots \\ c_{21} & c_{22} & c_{23} & \ldots \\ c_{31} & c_{32} & c_{33} & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \in \mathbb{F}_b^{\mathbb{N}\times\mathbb{N}}.$$

Example 4.78 For example, to generate a $(\mathbf{T}, 2)$-sequence over $\mathbb{Z}_2$ take the matrices

$$C_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \in \mathbb{Z}_2^{\mathbb{N}\times\mathbb{N}}$$

and

$$C_2 = \begin{pmatrix} \binom{0}{0} & \binom{1}{0} & \binom{2}{0} & \binom{3}{0} & \cdots \\ 0 & \binom{1}{1} & \binom{2}{1} & \binom{3}{1} & \cdots \\ 0 & 0 & \binom{2}{2} & \binom{3}{2} & \cdots \\ 0 & 0 & 0 & \binom{3}{3} & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ \vdots & \vdots & & \ddots & \ddots \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \dots \\ 0 & 1 & 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & & \ddots & \ddots & \ddots \end{pmatrix} \in \mathbb{Z}_2^{\mathbb{N}\times\mathbb{N}},$$

where the binomial coefficients are taken modulo 2.

To generate now one of the points $\boldsymbol{x}_n = (x_{n,1}, \dots, x_{n,s})$, with $n \in \mathbb{N}_0$, of the $(\mathbf{T}, s)$-sequence, we first write n in its b-adic (i.e., base b) expansion $n = \sum_{i=0}^{\infty} a_i b^i$ with $a_i \in \{0, \dots, b-1\}$ and $a_i = 0$ for all i large enough. Then take the column vector

$$\mathbf{n} = \begin{pmatrix} \varphi(a_0) \\ \varphi(a_1) \\ \varphi(a_2) \\ \vdots \end{pmatrix} \in (\mathbb{F}_b^{\mathbb{N}})^{\top}.$$

For example, to generate the point $\boldsymbol{x}_{13} = (x_{13,1}, x_{13,2})$ of a $(\mathbf{T}, s)$-sequence over $\mathbb{Z}_2$, write

$$n = 13 = 1 \cdot 2^0 + 0 \cdot 2^1 + 1 \cdot 2^2 + 1 \cdot 2^3 + 0 \cdot 2^4 + \cdots .$$

This corresponds to the vector

$$\mathbf{n} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}.$$

To generate the point $\boldsymbol{x}_n = (x_{n,1}, \dots, x_{n,s})$ we explain how to generate the ith coordinate $x_{n,i}$. The value of $x_{n,i}$ is obtained by multiplying the ith matrix C_i by $\mathbf{n}$ in $\mathbb{F}_b^m$, which gives as result a column vector over $\mathbb{F}_b$, say

$$C_i \mathbf{n} = \begin{pmatrix} \overline{y}_{n,i,1} \\ \overline{y}_{n,i,2} \\ \vdots \end{pmatrix} \in (\mathbb{F}_b^{\mathbb{N}})^{\top}.$$

(Note that for the multiplication only finitely many of the entries of $\mathbf{n}$ are different from zero, as we assumed that $\varphi(0) = \overline{0}$.) The elements $\varphi^{-1}(\overline{y}_{n,i,j}) \in \{0, \dots, b-1\}$, for $j \in \mathbb{N}$, are now the b-adic digits of $x_{n,i}$, i.e.,

$$x_{n,i} = \frac{\varphi^{-1}(\overline{y}_{n,i,1})}{b} + \frac{\varphi^{-1}(\overline{y}_{n,i,2})}{b^2} + \cdots .$$

Definition 4.79 We call the sequence $(\boldsymbol{x}_0, \boldsymbol{x}_1, \dots)$ constructed in this way a *digital sequence over* $\mathbb{F}_b$ *with generating matrices* $C_1, \dots, C_s$, or short, a *digital sequence*.

Since any sequence in $[0,1)^s$ is a $(\mathbf{T}, s)$-sequence in base b with a certain quality function $\mathbf{T}$ (at least for $\mathbf{T}(m) = m$) we also speak of a *digital* $(\mathbf{T}, s)$-*sequence over* $\mathbb{F}_b$.

Example 4.80 In Example 4.78, for $n = 13$, we have

$$C_1\mathbf{n} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

$$C_2\mathbf{n} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \dots \\ 0 & 1 & 0 & 1 & 0 & 1 & \dots \\ 0 & 0 & 1 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix},$$

which yields $x_{13,1} = \frac{1}{2} + \frac{1}{8} + \frac{1}{16} = \frac{11}{16}$, $x_{13,2} = \frac{1}{2} + \frac{1}{4} + \frac{1}{16} = \frac{13}{16}$, and hence $\boldsymbol{x}_{13} = \left(\frac{11}{16}, \frac{13}{16}\right)$.

Remark 4.81 Depending on the matrices $C_1, \dots, C_s$ it may happen that the vector $C_i\mathbf{n} =: \mathbf{y}_n$ contains infinitely many entries different from zero. For practical purposes this requires an adaptation of the point generation. Usually the vector $\mathbf{y}_n$ is truncated at a suitable place.

Further, another theoretical problem may arise. It should be avoided that the vector $\mathbf{y}_n$ contains only finitely many elements different from $\varphi(b-1)$. Because of the nonuniqueness of representation of the b-adic real numbers

represented by such "digit vectors", the net structure of the sequence in consideration would be destroyed. This does not happen for matrices like in the example above. But it may happen for other choices of matrices. This is the reason for the following additional condition on the matrices $C_1, \ldots, C_s$. Let

$$C_i = (c_{j,r}^{(i)})_{j,r \in \mathbb{N}} \in \mathbb{F}_b^{\mathbb{N} \times \mathbb{N}}$$

for $1 \le i \le s$. We demand that for all i and r we have $c_{j,r}^{(i)} = 0$ for all j large enough.

The quality function of digital sequences

As already mentioned, every sequence in $[0, 1)^s$ is a $(\mathbf{T}, s)$-sequence in base b, with $\mathbf{T}(m) = m$. Therefore we may ask: What is the strict quality function of the above $(\mathbf{T}, s)$-sequence in base b? The answer is given with the help of the following quantity ρ_m, which in some sense "measures" the "linear independence" of the s infinite matrices $C_1, \ldots, C_s$.

Definition 4.82 Let $C_1, \ldots, C_s$ be $\mathbb{N} \times \mathbb{N}$ matrices over the finite field $\mathbb{F}_b$. For any integers $1 \le i \le s$ and $m \ge 1$ by $C_i^{(m)}$ we denote the left upper $m \times m$ sub-matrix of C_i. Then

$$\rho_m = \rho_m(C_1, \ldots, C_s) := \rho(C_1^{(m)}, \ldots, C_s^{(m)}),$$

where ρ is the linear independence parameter defined for s-tuples of $m \times m$ matrices over $\mathbb{F}_b$ in Definition 4.50.

Example 4.83 In Example 4.78 above, for every $m \ge 1$ the matrices $C_1^{(m)}$ and $C_2^{(m)}$ are just the first and the third matrix of Example 4.57. For these matrices the value of ρ always equals m. Hence $\rho_m(C_1, C_2) = m$ for all $m \in \mathbb{N}$.

Now we can determine the strict quality function $\mathbf{T}$ of a digital sequence over $\mathbb{F}_b$. The proof of the following theorem gives some additional insight into the structure of a digital $(\mathbf{T}, s)$-sequence.

Theorem 4.84 *Let b be a prime power and let $\varphi : \{0, \ldots, b-1\} \to \mathbb{F}_b$ be a bijection with $\varphi(0) = \overline{0}$. The sequence $(\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots)$ constructed by the digital method with the $\mathbb{N} \times \mathbb{N}$ matrices $C_1, \ldots, C_s$ over $\mathbb{F}_b$ is a strict $(\mathbf{T}, s)$-sequence in base b with $\mathbf{T}(m) = m - \rho_m$ for all $m \in \mathbb{N}$, where ρ_m is the quantity defined in Definition 4.82.*

Proof By the definition of a $(\mathbf{T}, s)$-sequence we have to show that for any $m \in \mathbb{N}$ and any $k \in \mathbb{N}_0$ the point set

$$\{\boldsymbol{x}_{kb^m}, \ldots, \boldsymbol{x}_{kb^m+b^m-1}\}$$

is a strict $(\mathbf{T}(m), m, s)$-net in base b. (In fact, it suffices to show the strictness for at least one of these blocks.) Indeed, for given k and m, and any l between 0 and $b^m - 1$ let $k = \kappa_{r+1}b^r + \cdots + \kappa_1$ and $l = \lambda_{m-1}b^{m-1} + \cdots + \lambda_0$ be the base b representations of k and l. For $n = kb^m + l$ we have

$$\mathbf{n} = (\varphi(\lambda_0), \ldots, \varphi(\lambda_{m-1}), \varphi(\kappa_1), \ldots, \varphi(\kappa_{r+1}), \ldots)^\top \in (\mathbb{F}_b^{\mathbb{N}})^\top$$

and with the following representation of the matrices C_i,

$$C_i = \left(\begin{array}{c|c} C_i^{(m)} & D_i^{(m)} \\ \hline \multicolumn{2}{c}{F_i^{(m)}} \end{array}\right) \in \mathbb{F}_b^{\mathbb{N}\times\mathbb{N}}$$

we have

$$C_i\mathbf{n} = \left(\begin{array}{c} C_i^{(m)} \begin{pmatrix} \varphi(\lambda_0) \\ \vdots \\ \varphi(\lambda_{m-1}) \end{pmatrix} \\ \overline{0} \\ \overline{0} \\ \vdots \end{array}\right) + \left(\begin{array}{c} D_i^{(m)} \begin{pmatrix} \varphi(\kappa_1) \\ \vdots \\ \varphi(\kappa_{r+1}) \\ \overline{0} \\ \vdots \end{pmatrix} \\ \overline{0} \\ \overline{0} \\ \vdots \end{array}\right) + \begin{pmatrix} \overline{0} \\ \vdots \\ \overline{0} \\ F_i^{(m)}\mathbf{n} \end{pmatrix}.$$

Now we invoke Lemma 4.63 and Lemma 4.67 from Section 4.4. For the point

set under consideration the vector

$$\begin{pmatrix} & \begin{pmatrix} \varphi(\kappa_1) \\ \vdots \\ \varphi(\kappa_{r+1}) \\ \overline{0} \\ \vdots \end{pmatrix} \\ D_i^{(m)} & \overline{0} \\ & \overline{0} \\ & \vdots \end{pmatrix}$$

is constant. The term

$$\begin{pmatrix} \overline{0} \\ \vdots \\ \overline{0} \\ F_i^{(m)}\mathbf{n} \end{pmatrix}$$

increases the value of each coordinate of the point $\boldsymbol{x}_n$ by a value less than b^{-m} (here we use the additional condition in the definition of digital sequences). Therefore the point set

$$\{\boldsymbol{x}_{kb^m}, \ldots, \boldsymbol{x}_{kb^m+b^m-1}\}$$

is the digital net over $\mathbb{F}_b$ generated by the matrices $C_1^{(m)}, \ldots, C_s^{(m)}$, which is shifted by a digital shift of depth m. Hence, by Lemmas 4.52, 4.63, and 4.67, this point set is a strict (t, m, s)-net in base b with quality parameter t equal to the quality parameter of the digital net over $\mathbb{F}_b$ generated by the matrices $C_1^{(m)}, \ldots, C_s^{(m)}$. This parameter, by Lemma 4.52 and Definition 4.82, is $m - \rho_m$ and the result follows. □

Example 4.85 According to Example 4.83 and Theorem 4.84, the digital sequence from Example 4.78 provides a digital $(0, 2)$-sequence over $\mathbb{Z}_2$.

Distribution properties of digital sequences

Concerning the uniform distribution of a strict digital $(\mathbf{T}, s)$-sequence over $\mathbb{F}_b$, by Theorem 4.32 in Section 4.3, we have again that it is uniformly distributed if $\lim_{m\to\infty} m - \mathbf{T}(m) = \infty$. In contrast to the general case however, for digital $(\mathbf{T}, s)$-sequences, this condition can be shown to be a necessary and sufficient one.

Theorem 4.86 *Let b be a prime power. A strict digital $(\mathbf{T}, s)$-sequence over $\mathbb{F}_b$ is uniformly distributed modulo one, if and only if*

$$\lim_{m\to\infty} m - \mathbf{T}(m) = \infty.$$

For the proof of this result we need the following lemma.

Lemma 4.87 *Let b be a prime power. For integers $m \geq 1$ and t with $0 \leq t < m$, let $\mathbf{c}_1, \ldots, \mathbf{c}_{m-t} \in \mathbb{F}_b^m$ be given. Let L be the number of solutions of the system of linear equations $\mathbf{c}_j\mathbf{z} = 0$ for $1 \leq j \leq m-t$ in unknowns $\mathbf{z} \in (\mathbb{F}_b^m)^\top$. Then b^t divides L.*

Proof Let us consider the additive group of $(\mathbb{F}_b^{m-t})^\top$ and denote it by G. Let $\widehat{G}$ denote the dual group of characters χ of G. Let $\mathbf{c}_j := (c_{j,1}, \ldots, c_{j,m}) \in \mathbb{F}_b^m$ and

$$\mathbf{a}_i := \begin{pmatrix} c_{1,i} \\ \vdots \\ c_{m-t,i} \end{pmatrix} \in (\mathbb{F}_b^{m-t})^\top$$

for $1 \leq i \leq m$. Let H be the subgroup

$$H = \{z_1\mathbf{a}_1 + \cdots + z_m\mathbf{a}_m \,:\, z_1, \ldots, z_m \in \mathbb{F}_b\}$$

in G.

A character $\chi \in \widehat{G}$ is trivial on H if and only if it is trivial on all of the groups $H_i := \{z\mathbf{a}_i \,:\, z \in \mathbb{F}_b\}$.

We have

$$\begin{aligned} L &= \sum_{\substack{\mathbf{z}\in(\mathbb{F}_b^m)^\top \\ \mathbf{c}_j\mathbf{z}=0 \\ \forall\, j\in\{1,\ldots,m-t\}}} 1 \\ &= \sum_{\mathbf{z}\in(\mathbb{F}_b^m)^\top} \frac{1}{b^{m-t}} \sum_{\chi\in\widehat{G}} \chi\begin{pmatrix} \mathbf{c}_1\mathbf{z} \\ \vdots \\ \mathbf{c}_{m-t}\mathbf{z} \end{pmatrix} \\ &= \frac{1}{b^{m-t}} \sum_{\chi\in\widehat{G}} \sum_{\mathbf{z}\in(\mathbb{F}_b^m)^\top} \chi\begin{pmatrix} \mathbf{c}_1\mathbf{z} \\ \vdots \\ \mathbf{c}_{m-t}\mathbf{z} \end{pmatrix} \\ &= \frac{1}{b^{m-t}} \sum_{\chi\in\widehat{G}} \prod_{i=1}^{m} \sum_{z_i\in\mathbb{F}_b} \chi(z_i\mathbf{a}_i). \end{aligned}$$

Now

$$\sum_{z_i \in \mathbb{F}_b} \chi(z_i \mathbf{a}_i) = \begin{cases} b & \text{if } \chi \text{ is trivial on } H_i, \\ 0 & \text{otherwise,} \end{cases}$$

thus

$$\prod_{i=1}^{m} \sum_{z_i \in \mathbb{F}_b} \chi(z_i \mathbf{a}_i) = \begin{cases} b^m & \text{if } \chi \text{ is trivial on } H, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently

$$L = \frac{1}{b^{m-t}} \sum_{\substack{\chi \in \widehat{G} \\ \chi \text{ trivial on } H}} b^m = b^t \, |\{\chi \in \widehat{G} : \chi \text{ trivial on } H\}|$$

and hence b^t divides L. □

Proof of Theorem 4.86 By Theorem 4.32 it suffices to show that the digital sequence $\mathcal{S} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots)$ is not uniformly distributed if $m - \mathbf{T}(m)$ does not tend to infinity. Since $m - \mathbf{T}(m)$ is monotonically increasing, this means $m - \mathbf{T}(m) = \kappa$ for some integer $\kappa \geq 0$ and all $m \geq m_0$, for some integer $m_0 \geq 0$. Hence, for all $m \geq m_0$, the point set $\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{b^m-1}\}$ is a strict $(m - \kappa, m, s)$-net in base b, in particular, it is never a $(m - \kappa - 1, m, s)$-net in base b. Assume that $\mathcal{S}$ is generated by the $\mathbb{N} \times \mathbb{N}$ matrices $C_1, \ldots, C_s$. Let $\mathbf{c}_j^{(i)}$ be the jth row vector of the ith matrix and for $m \in \mathbb{N}$ let $\pi_m(\mathbf{c}_j^{(i)})$ be the vector from $\mathbb{F}_b^m$ consisting of the first m components of $\mathbf{c}_j^{(i)}$. Hence, there are integers $d_1, \ldots, d_s \geq 0$, which may depend on m, i.e., $d_i = d_i(m)$, with $d_1 + \cdots + d_s = \kappa + 1$ and elements $\overline{e}_j^{(i)} \in \mathbb{F}_b$, $1 \leq j \leq d_i$ and $1 \leq i \leq s$ such that the system

$$\pi_m(\mathbf{c}_j^{(i)})\mathbf{n} = \overline{e}_j^{(i)} \text{ for } 1 \leq j \leq d_i \text{ and } 1 \leq i \leq s$$

has $L \neq b^{m-\kappa-1}$ solutions $\mathbf{n} \in (\mathbb{F}_b^m)^\top$ (see Lemma 4.59 and the proof of Theorem 4.84). Hence, the corresponding homogeneous system of equations

$$\pi_m(\mathbf{c}_j^{(i)})\mathbf{n} = \overline{0} \text{ for } 1 \leq j \leq d_i \text{ and } 1 \leq i \leq s$$

has more than $b^{m-\kappa-1}$ solutions, indeed, by Lemma 4.87, at least $2b^{m-\kappa-1}$ solutions. Therefore the box

$$J = J(m) = \prod_{i=1}^{s} \left[0, \frac{1}{b^{d_i}}\right)$$

of volume $b^{-\kappa-1}$ contains at least $2b^{m-\kappa-1}$ points (see again the proof of Theorem 4.84). As there is only a finite number of boxes $J(m)$, there is one

box, say J^*, such that for infinitely many $m \geq m_0$ we have $J(m) = J^*$. Therefore we obtain

$$\left| \frac{A(J^*, b^m, \mathcal{S})}{b^m} - \lambda_s(J^*) \right| \geq \frac{1}{b^{\kappa+1}}$$

for infinitely many $m \geq m_0$. Thus $\mathcal{S} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots)$ is not uniformly distributed modulo one. □

Corollary 4.88 *Let b be a prime power. The $\mathbb{N} \times \mathbb{N}$ matrices $C_1, \ldots, C_s$ over $\mathbb{F}_b$ generate a uniformly distributed sequence in $[0,1)^s$ if and only if*

$$\lim_{m \to \infty} \rho_m = \infty,$$

where $\rho_m = \rho(C_1^{(m)}, \ldots, C_s^{(m)})$ is the independence quantity defined in Definition 4.82.

Proof This follows from Theorem 4.84 and from Theorem 4.86. □

Remark 4.89 From the above result and from Theorem 4.34 it also follows that a strict digital $(\mathbf{T}, s)$-sequence over $\mathbb{F}_b$ is even well-distributed if and only if $\lim_{m \to \infty} m - \mathbf{T}(m) = \infty$.

Propagation rules for digital sequences

For digital $(\mathbf{T}, s)$-sequences we have the following simple propagation rules:

1. Any digital $(\mathbf{T}, s)$-sequence over $\mathbb{F}_b$ is a digital $(\mathbf{U}, s)$-sequence over $\mathbb{F}_b$ for all $\mathbf{U}$ with $\mathbf{U}(m) \geq \mathbf{T}(m)$ for all m.
2. If the matrices $C_1, \ldots, C_s$ generate a digital $(\mathbf{T}, s)$-sequence over $\mathbb{F}_b$ and if we take any s' (where $s' \leq s$) of these matrices, then these matrices form a digital $(\mathbf{T}, s')$-sequence over $\mathbb{F}_b$.

Structural results for digital sequences

Note that there is no analogue to Lemma 4.63 for digital $(\mathbf{T}, s)$-sequences over $\mathbb{F}_b$. In general, common addition of a fixed (even very "small") constant vector can disturb the $(\mathbf{T}, s)$-sequence property (i.e., can destroy its quality). However, digitally shifting using $\oplus_{b,\varphi}$ from Definition 4.65 is possible.

Definition 4.90 Let b be a prime power and let $\varphi : \{0, \ldots, b-1\} \to \mathbb{F}_b$ be a bijection with $\varphi(0) = \overline{0}$. For a sequence $\mathcal{S} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots)$ in $[0,1)^s$ and a $\boldsymbol{\sigma} \in [0,1)^s$ the sequence $\mathcal{S}_{\boldsymbol{\sigma}} = (\boldsymbol{x}_0 \oplus \boldsymbol{\sigma}, \boldsymbol{x}_1 \oplus \boldsymbol{\sigma}, \ldots\}$ is called the (b-adic) *digitally shifted sequence* $\mathcal{S}$, or the (b-adic) *digitally shifted version of* $\mathcal{S}$. The vector $\boldsymbol{\sigma} \in [0,1)^s$ is called a (b-adic) *digital shift*.

If we use a digital shift in conjunction with a $(\mathbf{T}, s)$-sequence, then they are always considered to be in the same base b and with the same bijection φ. Therefore, if it is clear with respect to which base b a point is shifted we may omit the phrase "b-adic".

We show that a digital shift preserves the $(\mathbf{T}, s)$-sequence structure.

Lemma 4.91 *Let b be a prime power and let $\varphi : \{0, \ldots, b-1\} \to \mathbb{F}_b$ be a bijection with $\varphi(0) = \overline{0}$. Let $\mathcal{S} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots)$ be a (strict) digital $(\mathbf{T}, s)$-sequence over $\mathbb{F}_b$ and let $\boldsymbol{\sigma} \in [0,1)^s$. Then the digitally shifted sequence $\mathcal{S}_{\boldsymbol{\sigma}} = (\boldsymbol{y}_0, \boldsymbol{y}_1, \ldots)$ is a (strict) $(\mathbf{T}, s)$-sequence in base b.*

Remark 4.92 For $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_s) \in [0,1)^s$ it should be avoided that the σ_i, $1 \le i \le s$, contain only finitely many b-adic digits different from $\varphi^{-1}(b-1)$ (see also the Remark 4.81).

Proof of Lemma 4.91 We use the notation of the proof of Theorem 4.84. For $1 \le i \le s$ and $\sigma_i = \frac{\varsigma_{i,1}}{b} + \frac{\varsigma_{i,2}}{b^2} + \cdots$ with $\varsigma_{i,k} \in \{0, \ldots, b-1\}$ for $k \ge 1$, let $\boldsymbol{\sigma}_i := (\varphi(\varsigma_{i,1}), \varphi(\varsigma_{i,2}), \ldots)^\top \in (\mathbb{F}_b^\infty)^\top$.

The subsequence

$$\{\boldsymbol{y}_{kb^m}, \ldots, \boldsymbol{y}_{kb^m+b^m-1}\}$$

is obtained by calculating, for

$$\mathbf{n} = (\varphi(\lambda_0), \ldots, \varphi(\lambda_{m-1}), \varphi(\kappa_1), \ldots, \varphi(\kappa_{r+1}), \overline{0}, \ldots)^\top \in (\mathbb{F}_b^{\mathbb{N}})^\top,$$

the shifted vector

$$C_i\mathbf{n} + \boldsymbol{\sigma}_i = \begin{pmatrix} C_i^{(m)} \begin{pmatrix} \varphi(\lambda_0) \\ \vdots \\ \varphi(\lambda_{m-1}) \end{pmatrix} \\ \overline{0} \\ \overline{0} \\ \vdots \end{pmatrix} + \begin{pmatrix} \varphi(\varsigma_{i,1}) \\ \vdots \\ \varphi(\varsigma_{i,m}) \\ \overline{0} \\ \vdots \end{pmatrix}$$

$$+ \begin{pmatrix} D_i^{(m)} \begin{pmatrix} \varphi(\kappa_1) \\ \vdots \\ \varphi(\kappa_{r+1}) \\ \overline{0} \\ \vdots \end{pmatrix} \\ \overline{0} \\ \overline{0} \\ \vdots \end{pmatrix} + \begin{pmatrix} \overline{0} \\ \vdots \\ \overline{0} \\ F_i^{(m)}\mathbf{n} + \begin{pmatrix} \varphi(\varsigma_{i,m+1}) \\ \varphi(\varsigma_{i,m+2}) \\ \vdots \end{pmatrix} \end{pmatrix}.$$

Using Lemma 4.63 and Lemma 4.67, this yields a (strict) $(\mathbf{T}(m), m, s)$-net in base b and the result follows. □

Like digital (t, m, s)-nets over $\mathbb{F}_b$, also digital $(\mathbf{T}, s)$-sequences over $\mathbb{F}_b$ have a group structure. Recall that $([0,1)^s, \oplus)$ is an abelian group.

Theorem 4.93 *Let b be a prime power and let $\varphi : \{0, \ldots, b-1\} \to \mathbb{F}_b$ be a bijection with $\varphi(0) = \overline{0}$. Any digital $(\mathbf{T}, s)$-sequence over $\mathbb{F}_b$ is a subgroup of $([0,1)^s, \oplus)$. If the points of the digital sequence are pairwise different, then this subgroup is isomorphic to the additive group $\widetilde{\mathbb{F}}_b^{\mathbb{N}} := \{(g_1, g_2, \ldots) \in \mathbb{F}_b^{\mathbb{N}} : g_i = \overline{0}$ for almost all $i \in \mathbb{N}\}$.*

Proof Any nonnegative integer n is uniquely represented by a vector

$$\mathbf{n} = \begin{pmatrix} \overline{n}_0 \\ \overline{n}_1 \\ \vdots \end{pmatrix} \in (\widetilde{\mathbb{F}}_b^{\mathbb{N}})^\top,$$

where $n := n_0 + n_1 b + \cdots$ via $n_i = \varphi^{-1}(\overline{n}_i)$, and to any such integer belongs an element $\boldsymbol{x}_n$ of the digital sequence. This holds also the other way round, namely, to any point $\boldsymbol{x}_n$ from the digital sequence belongs a unique vector $\mathbf{n} \in (\widetilde{\mathbb{F}}_b^{\mathbb{N}})^\top$ and therefore a uniquely determined nonnegative integer n. Hence the mapping

$$\Psi : (\widetilde{\mathbb{F}}_b^{\mathbb{N}})^\top \to \{\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots\},\ \mathbf{n} \mapsto \boldsymbol{x}_n$$

is bijective. It can be shown, like in the proof of Lemma 4.72, that Ψ is a group homomorphism. Hence the result follows. □

Exercises

4.1 Construct "by hand" a $(0, 2, 2)$-net in base 3.

4.2 Let $b \geq 2$ be an integer. Show that for any $s \geq 2$ and any $m \geq 2$ there is a $(m-1, m, s)$-net in base 2.

4.3 Show that the van der Corput sequence in base b is a $(0, 1)$-sequence in base b.

4.4 Show that the 4×4 matrices

$$C_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ C_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$C_3 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; C_4 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

over $\mathbb{Z}_2$ generate a digital $(1, 4, 4)$-net over $\mathbb{Z}_2$.

4.5 Determine the strict quality parameter t of the digital $(t, 3, 2)$-net over $\mathbb{F}_4$ from Example 4.49.

4.6 Prove Lemma 4.59.

4.7 Prove Lemma 4.61.

4.8 Let b be a prime power, let $s \in \mathbb{N}$ and let $C_1, \ldots, C_s$ be $\mathbb{N} \times \mathbb{N}$ matrices over $\mathbb{F}_b$. For $m \in \mathbb{N}$ let $C_i^{(m)}$ be the left upper $m \times m$ submatrix of C_i, $1 \le i \le s$. Show that $C_1, \ldots, C_s$ generate a strict digital $(\mathbf{T}, s)$-sequence over $\mathbb{F}_b$ with $\mathbf{T}(m) = \delta(C_1^{(m)}, \ldots, C_s^{(m)})$, where δ is defined as in Definition 4.58.

4.9 Show that a (strict) digital (t, m, s)-net in base b, which is shifted by a digital shift of depth m (Definition 4.69) independently in each coordinate, is, with probability one, a (strict) (t, m, s)-net in base b with the same quality parameter t. (Assume that the shifts are uniformly and i.i.d..)

4.10 Show that a (strict) digital (t, m, s)-net in base b, which is shifted by a simplified digital shift (Definition 4.70), independently in each coordinate, is again a (strict) (t, m, s)-net in base b with the same quality parameter t.

4.11 Let b be a prime power and let the $\mathbb{N} \times \mathbb{N}$ matrices $C_1, \ldots, C_s$ generate a digital $(\mathbf{T}, s)$-sequence over the finite field $\mathbb{F}_b$. For any $m \ge 1$ consider the left upper $m \times m$ sub-matrices $C_1^{(m)}, \ldots, C_s^{(m)}$. Take

$$C_{s+1}^{(m)} := E_m' = \begin{pmatrix} \overline{0} & \overline{0} & \ldots & \overline{0} & \overline{1} \\ \overline{0} & & \iddots & \overline{1} & \overline{0} \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ \overline{0} & \overline{1} & \iddots & & \overline{0} \\ \overline{1} & \overline{0} & \ldots & \overline{0} & \overline{0} \end{pmatrix} \in \mathbb{F}_b^{m \times m}.$$

Show that the $m \times m$ matrices $C_1^{(m)}, \ldots, C_s^{(m)}, C_{s+1}^{(m)}$ generate a digital $(r(m), m, s+1)$-net over $\mathbb{F}_b$ with $r(m) := \max\{\mathbf{T}(0), \ldots, \mathbf{T}(m)\}$. *Remark:* Note that this is a "digital version" of Lemma 4.38. Note also the increase of the dimension from s to $s+1$.

4.12 For $k \in \mathbb{N}$ with b-adic expansion $k = \kappa_0 + \kappa_1 b + \cdots + \kappa_{a-1} b^{a-1}$, where $\kappa_{a-1} \ne 0$, we define $\rho(k) = a$. Furthermore we define $\rho(0) = 0$. For

$\boldsymbol{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ let $\rho(\boldsymbol{k}) = \sum_{i=1}^s \rho(k_i)$. (This weight function is intimately related to the so-called NRT-weight which is introduced and used in Chapter 7 (Definition 7.1). See also Chapter 16, Definition 16.24.)

Show that if a point set $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{b^m-1}\}$ consisting of b^m points in $[0,1)^s$ is a (t,m,s)-net in base b, $b \geq 2$ an arbitrary integer, then we have

$$\sum_{n=0}^{b^m-1} {}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_n) = 0 \quad \text{for all } \boldsymbol{k} \in \mathbb{N}_0^s \setminus \{\boldsymbol{0}\} \text{ with } 0 < \rho(\boldsymbol{k}) \leq m-t.$$

Remark: This is [96, Lemma 1]. Compare with Lemma 4.75, but note that here $\mathcal{P}$ does not need to be a digital net. *Hint:* Show that the Walsh function ${}_b\mathrm{wal}_{\boldsymbol{k}}$ for $\boldsymbol{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ satisfying $0 \leq k_i < b^{g_i}$ for $1 \leq i \leq s$ can be written as a step function of the form ${}_b\mathrm{wal}_{\boldsymbol{k}} = \sum_{\boldsymbol{a}} c_{\boldsymbol{a}} \chi_{J_{\boldsymbol{a}}}$ with coefficients $c_{\boldsymbol{a}} \in \mathbb{R}$, where $J_{\boldsymbol{a}} = \prod_{i=1}^s [a_i b^{-g_i}, (a_i+1)b^{-g_i})$ and where the summation is overall possible $\boldsymbol{a} = (a_1, \ldots, a_s) \in \mathbb{N}_0^s$ with $0 \leq a_i < b^{g_i}$ for $1 \leq i \leq s$. Show that $\sum_{\boldsymbol{a}} c_{\boldsymbol{a}} = 0$ whenever $\boldsymbol{k} \neq \boldsymbol{0}$ and use the (t,m,s)-net property of $\mathcal{P}$.

4.13 Show the converse of Exercise 4.12. If $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{b^m-1}\}$ is a point set consisting of b^m points in $[0,1)^s$ such that

$$\sum_{n=0}^{b^m-1} {}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_n) = 0 \quad \text{for all } \boldsymbol{k} \in \mathbb{N}_0^s \setminus \{\boldsymbol{0}\} \text{ with } 0 < \rho(\boldsymbol{k}) \leq m-t,$$

then $\mathcal{P}$ is a (t,m,s)-net in base b. *Remark:* This is [96, Lemma 2]. Note that $\mathcal{P}$ is in general not a digital net. *Hint:* Consider the Walsh series expansion of the characteristic function of an arbitrary elementary b-adic elementary interval of order $m-t$ and use Lemma 3.9.

4.14 Show that for the b-adic spectral test (see Exercise 3.8) of a (t,m,s)-net $\mathcal{P}$ in base b we have $\sigma_{b,b^m}(\mathcal{P}) \leq b^{t-m-1}$. *Remark:* This is [96, Theorem 4].

4.15 Show that for the b-adic spectral test of a strict digital (t,m,s)-net $\mathcal{P}$ in base b we have $\sigma_{b,b^m}(\mathcal{P}) = b^{t-m-1}$. *Remark:* This is [96, Corollary 8].

5

Discrepancy estimates and average type results

The motivation for introducing and studying the concept of (t, m, s)-nets and $(\mathbf{T}, s)$-sequences was to generate point sets (also sometimes in high dimensions) with discrepancy as small as possible. In this chapter we give an overview on theoretical results for the discrepancy of (digital) nets and sequences.

While singular results were already given by Sobol′ [251] and by Faure [66], a first systematic study of the discrepancy of nets was given by Niederreiter [170]. These results can also be found in [175, Chapter 4]. Further results on the star discrepancy of digital nets and sequences, mainly for low dimensions, can be found in [40, 69, 70, 72, 123, 124, 142, 143, 211].

After the work of Niederreiter in [170] and [175], metrical and average results on the discrepancy of nets and net-sequences were given, see, for instance, [132, 133, 134, 136, 138]. Further, also the study of weighted discrepancy of net-type point sets received considerable attention in recent years (see, for example, [49, 144]).

Even though we have many results for the extreme and star discrepancy, very little is known about concrete theoretical estimates for the L_p-discrepancy, especially for net-type point sets. Singular results in this direction can be found in [20, 22, 73, 140, 141, 210, 242] (results concerning the L_2-discrepancy are presented in Chapter 16).

The aims of this chapter are the following:

1. We illustrate the ideas underlying all discrepancy estimates for (t, m, s)-nets with help of detailed elaborated and illustrated proofs of a few discrepancy results.
2. We give a collection of concrete discrepancy estimates for net-type point sets with references for their proofs.

3. We give a discussion on these results concerning their value for applications.
4. We give a collection of metrical and average type estimates for the quality parameter and the discrepancy of net-type point sets.

5.1 Discrepancy estimates for (t, m, s)-nets and $(\mathbf{T}, s)$-sequences

In this section we give a collection of concrete star discrepancy estimates for net-type point sets and sequences and we illustrate the ideas underlying all such estimates.

Star discrepancy estimates for (t, m, s)-nets

The most important and general applicable concrete discrepancy estimates for (t, m, s)-nets are the following two results given by Niederreiter in [175, Theorem 4.5 and Theorem 4.6].

Theorem 5.1 *The star discrepancy of a (t, m, s)-net $\mathcal{P}$ in base $b \geq 3$ satisfies*

$$b^m D^*_{b^m}(\mathcal{P}) \leq b^t \sum_{i=0}^{s-1} \binom{s-1}{i} \binom{m-t}{i} \left\lfloor \frac{b}{2} \right\rfloor^i.$$

Theorem 5.2 *The star discrepancy of a (t, m, s)-net $\mathcal{P}$ in an even base b satisfies*

$$b^m D^*_{b^m}(\mathcal{P}) \leq b^t \sum_{i=0}^{s-1} \binom{m-t}{i} \left(\frac{b}{2}\right)^i + \left(\frac{b}{2} - 1\right) b^t \sum_{i=0}^{s-2} \binom{m-t+i+1}{i} \left(\frac{b}{2}\right)^i.$$

For applications the case $b = 2$ is of importance. For this case we obtain the following corollary from the last result.

Corollary 5.3 *The star discrepancy of a (t, m, s)-net $\mathcal{P}$ in base $b = 2$ satisfies*

$$2^m D^*_{2^m}(\mathcal{P}) \leq 2^t \sum_{i=0}^{s-1} \binom{m-t}{i}.$$

Below we present a detailed and self-contained proof for this bound.

Both of the above theorems give results for even bases $b \geq 4$. For the special cases $s = 2, 3$, and 4 alternative estimates are given which in some cases give improvements of the results that can be derived from Theorem 5.1 for these cases.

The corresponding result for $s = 2$ was proved by Dick & Kritzer [41, Theorem 1] (see also [142, Theorem 5] for the special case of digital $(0,m,2)$-nets over $\mathbb{Z}_2$).

Theorem 5.4 *For $s = 2$, the star discrepancy of a (t,m,s)-net $\mathcal{P}$ in base b satisfies*

$$b^m D_{b^m}^*(\mathcal{P}) \le b^m D_{b^{m-t}}^*(\mathcal{H}_{b,m-t}) + b^t,$$

where $\mathcal{H}_{b,m-t}$ denotes the two-dimensional Hammersley point set in base b consisting of b^{m-t} points (see Definition 3.44). If $m - t \ge 2$ we obtain

$$b^m D_{b^m}^*(\mathcal{P}) \le b^t \left(\frac{b^2}{4(b+1)}(m-t) + \frac{9}{4} + \frac{1}{b} \right),$$

for even bases $b \ge 2$ and

$$b^m D_{b^m}^*(\mathcal{P}) \le b^t \left(\frac{b-1}{4}(m-t) + \frac{9}{4} + \frac{1}{b} \right),$$

for odd bases $b \ge 3$.

Remark 5.5 This result improves [175, Theorem 4.7], which states that for $s = 2$ the star discrepancy of a (t,m,s)-net $\mathcal{P}$ in base b satisfies

$$b^m D_{b^m}^*(\mathcal{P}) \le b^t \left\lfloor \frac{b-1}{2}(m-t) + \frac{3}{2} \right\rfloor.$$

Remark 5.6 It follows from Theorem 5.4 that among all $(0,m,2)$-nets in base b, the two-dimensional Hammersley point set in base b consisting of b^m points (which is of course itself a (digital) $(0,m,2)$-net in base b by Lemma 4.13) has, up to the term b^0, the worst star discrepancy.

The following result for nets in dimension $s = 3$ is [175, Theorem 4.8].

Theorem 5.7 *For $s = 3$, the star discrepancy of a (t,m,s)-net $\mathcal{P}$ in base b satisfies*

$$b^m D_{b^m}^*(\mathcal{P}) \le b^t \left\lfloor \left(\frac{b-1}{2} \right)^2 (m-t)^2 + \frac{b-1}{2}(m-t) + \frac{9}{4} \right\rfloor.$$

Remark 5.8 For digital $(0,m,3)$-nets $\mathcal{P}$ over $\mathbb{Z}_2$ we have the improvement $2^m D_{2^m}^*(\mathcal{P}) \le m^2/6 + O(m)$, by [211, Theorem 1].

The following result for nets in dimension $s = 4$ is [175, Theorem 4.9].

Theorem 5.9 *For $s = 4$, the star discrepancy of a (t,m,s)-net $\mathcal{P}$ in base b satisfies*

$b^m D_{b^m}^*(\mathcal{P})$

$$\leq b^t \left\lfloor \left(\frac{b-1}{2}\right)^3 (m-t)^3 + \frac{3(b-1)^2}{8}(m-t)^2 + \frac{3(b-1)}{8}(m-t) + \frac{15}{4} \right\rfloor .$$

These estimates are used by Niederreiter [175, Theorem 4.10] to obtain the following asymptotic result for the discrepancy of (t, m, s)-nets in base b.

Theorem 5.10 *The star discrepancy of a (t, m, s)-net $\mathcal{P}$ in base b with $m > 0$ satisfies*

$$b^m D^*_{b^m}(\mathcal{P}) \leq B(s, b) b^t m^{s-1} + O(b^t m^{s-2}), \tag{5.1}$$

where the implied O-constant depends only on b and s. Here $B(s, b) = \left(\frac{b-1}{2}\right)^{s-1}$ if either $s = 2$ or $b = 2$, $s = 3, 4$; otherwise $B(s, b) = \frac{\lfloor b/2 \rfloor^{s-1}}{(s-1)!}$.

Remark 5.11 Using the same method as Niederreiter, Kritzer [123] improved the values of $B(s, b)$ by a factor of roughly $1/2$.

Proof of Theorem 5.10 For the expression in Theorem 5.1, for large enough m we have

$$\begin{aligned}
& b^t \sum_{i=0}^{s-1} \binom{s-1}{i} \binom{m-t}{i} \left\lfloor \frac{b}{2} \right\rfloor^i \\
& \leq b^t \left\lfloor \frac{b}{2} \right\rfloor^{s-1} \binom{m-t}{s-1} + b^t \left\lfloor \frac{b}{2} \right\rfloor^{s-2} (m-t)^{s-2} 2^{s-1} \\
& \leq b^t \left\lfloor \frac{b}{2} \right\rfloor^{s-1} \frac{m^{s-1}}{(s-1)!} + O(b^t m^{s-2})
\end{aligned}$$

with an implied O-constant depending only on s and b. For the expression in Theorem 5.2, analogously, we have

$$\begin{aligned}
& b^t \sum_{i=0}^{s-1} \binom{m-t}{i} \left(\frac{b}{2}\right)^i + \left(\frac{b}{2} - 1\right) b^t \sum_{i=0}^{s-2} \binom{m-t+i+1}{i} \left(\frac{b}{2}\right)^i \\
& \leq b^t \binom{m-t}{s-1} \left(\frac{b}{2}\right)^{s-1} + b^t \left(\frac{b}{2}\right)^{s-2} (m-t)^{s-2} s \\
& \quad + \left(\frac{b}{2}\right)^{s-1} b^t \, s \, (m-t+s-1)^{s-2} \\
& \leq b^t \left(\frac{b}{2}\right)^{s-1} \frac{m^{s-1}}{(s-1)!} + O(b^t m^{s-2})
\end{aligned}$$

with an implied O-constant depending only on s and b. Hence the result

Appendix A
Walsh functions

Walsh functions play a very important role in the analysis of digital nets over $\mathbb{Z}_b$. In this Appendix we recall the definition of Walsh functions and we provide some important and useful results concerning these functions. Many of these results are used within this book without further comment. A standard reference for the theory of Walsh functions is the book of Schipp, Wade & Simon [230]. This overview here is mainly based on [214].

A.1 Definition of Walsh functions

In 1923 Walsh [258] introduced a system of functions which is in some way similar to the trigonometric function system $\{\mathrm{e}^{2\pi\mathrm{i}kx} : k \in \mathbb{Z}\}$ which is connected to the well known Fourier theory. (However, the differences will become clear in a moment.)

For $b \geq 2$ we denote by ω_b the primitive bth root of unity $\mathrm{e}^{2\pi\mathrm{i}/b}$.

Definition A.1 Let $k \in \mathbb{N}_0$ with b-adic expansion $k = \kappa_0 + \kappa_1 b + \kappa_2 b^2 + \cdots$ (this expansion is obviously finite). The *kth b-adic Walsh function* ${}_b\mathrm{wal}_k : \mathbb{R} \to \mathbb{C}$, periodic with period one, is defined as

$$ {}_b\mathrm{wal}_k(x) = \omega_b^{\kappa_0\xi_1 + \kappa_1\xi_2 + \kappa_2\xi_3 + \cdots}, $$

for $x \in [0,1)$ with b-adic expansion $x = \xi_1 b^{-1} + \xi_2 b^{-2} + \xi_3 b^{-3} + \cdots$ (unique in the sense that infinitely many of the digits ξ_i must be different from $b-1$).

We call the system $\{ {}_b\mathrm{wal}_k : k \in \mathbb{N}_0\}$ the *b-adic Walsh function system*.

In the literature the function system defined above is often called the generalised Walsh function system. Only in the case $b = 2$ one speaks of Walsh functions. However, within this book we also speak of Walsh functions in the more general b-adic case.

One of the main differences between Walsh functions and the trigonometric functions is that Walsh functions are only piecewise continuous. This is clear, since Walsh functions are step functions as we show now.

Let $k \in \mathbb{N}_0$ with b-adic expansion $k = \kappa_0 + \kappa_1 b + \cdots + \kappa_{r-1} b^{r-1}$. Let $J = [a/b^r, (a+1)/b^r)$, with an integer $0 \le a < b^r$, be a so-called *elementary b-adic interval of order r*. Let a have b-adic expansion of the form $a = \alpha_0 + \alpha_1 b + \cdots + \alpha_{r-1} b^{r-1}$. Then any $x \in J$ has b-adic expansion $x = \alpha_{r-1} b^{-1} + \alpha_{r-2} b^{-2} + \cdots + \alpha_0 b^{-r} + \xi_{r+1} b^{-(r+1)} + \xi_{r+2} b^{-(r+2)} + \cdots$ with some digits $0 \le \xi_i \le b-1$ for $i \ge r+1$ and hence

$$ {}_b\mathrm{wal}_k(x) = \omega_b^{\kappa_0 \alpha_{r-1} + \cdots + \kappa_{r-1} \alpha_0} = {}_b\mathrm{wal}_k(a/b^r). $$

We summarise this result in the following proposition.

Proposition A.2 *Let $k \in \mathbb{N}$ with $b^{r-1} \le k < b^r$. Then the kth Walsh function ${}_b\mathrm{wal}_k$ is constant on elementary b-adic intervals of order r of the form $[a/b^r, (a+1)/b^r)$ with value ${}_b\mathrm{wal}_k(a/b^r)$. Further, ${}_b\mathrm{wal}_0 = 1$ identical.*

Now we generalise the definition of Walsh functions to higher dimensions.

Definition A.3 For dimension $s \ge 2$, and $k_1, \ldots, k_s \in \mathbb{N}_0$ we define the *s-dimensional b-adic Walsh function* ${}_b\mathrm{wal}_{k_1,\ldots,k_s} : \mathbb{R}^s \to \mathbb{C}$ by

$$ {}_b\mathrm{wal}_{k_1,\ldots,k_s}(x_1, \ldots, x_s) := \prod_{j=1}^{s} {}_b\mathrm{wal}_{k_j}(x_j). $$

For vectors $\boldsymbol{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ and $\boldsymbol{x} = (x_1, \ldots, x_s) \in [0,1)^s$ we write, with some abuse of notation,

$$ {}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) := {}_b\mathrm{wal}_{k_1,\ldots,k_s}(x_1, \ldots, x_s). $$

The system $\{ {}_b\mathrm{wal}_{\boldsymbol{k}} : \boldsymbol{k} \in \mathbb{N}_0^s \}$ is called the *s-dimensional b-adic Walsh function system*.

As any s-dimensional Walsh function is a product of one-dimensional Walsh functions, it is clear that s-dimensional Walsh functions are step functions too.

A.2 Basic properties of Walsh functions

We introduce some notation. By $\oplus$ we denote the *digit-wise addition modulo b*, i.e., for $x = \sum_{i=w}^{\infty} \xi_i b^{-i}$ and $y = \sum_{i=w}^{\infty} \eta_i b^{-i}$ we define

$$ x \oplus y := \sum_{i=w}^{\infty} \zeta_i b^{-i}, \quad \text{where} \quad \zeta_i \equiv \xi_i + \eta_i \pmod{b}, $$

provided that infinitely many ζ_i are different from $b-1$. By $\ominus$ we denote the digit-wise subtraction modulo b, i.e.,

$$x \ominus y := \sum_{i=w}^{\infty} \zeta_i b^{-i}, \quad \text{where} \quad \zeta_i \equiv \xi_i - \eta_i \pmod{b},$$

provided that infinitely many ζ_i are different from $b-1$. Correspondingly, we define $\ominus x := 0 \ominus x$. For vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ we define $\boldsymbol{x} \oplus \boldsymbol{y}$, $\boldsymbol{x} \ominus \boldsymbol{y}$, and $\ominus \boldsymbol{x}$ component wise. Note that all these operations depend on the base b.

Proposition A.4 *For all $k, l \in \mathbb{N}_0$ we have*

$$ {}_b\mathrm{wal}_k \cdot {}_b\mathrm{wal}_l = {}_b\mathrm{wal}_{k \oplus l} \quad \textit{and} \quad \frac{1}{{}_b\mathrm{wal}_k} = \overline{{}_b\mathrm{wal}_k} = {}_b\mathrm{wal}_{\ominus k}.$$

Proof Let $k = \kappa_0 + \kappa_1 b + \kappa_2 b^2 + \cdots$ and $l = \lambda_0 + \lambda_1 b + \lambda_2 b^2 + \cdots$. Then we have

$$\begin{aligned} {}_b\mathrm{wal}_k(x)\, {}_b\mathrm{wal}_l(x) &= \omega_b^{\sum_{i\geq 0} \kappa_i \xi_{i+1}} \omega_b^{\sum_{i\geq 0} \lambda_i \xi_{i+1}} = \omega_b^{\sum_{i\geq 0} (\kappa_i + \lambda_i) \xi_{i+1}} \\ &= \omega_b^{\sum_{i\geq 0} (\kappa_i \oplus \lambda_i) \xi_{i+1}} = {}_b\mathrm{wal}_{k \oplus l}(x), \end{aligned}$$

where we used the periodicity of $z \mapsto \omega_b^z$, and also

$$\frac{1}{{}_b\mathrm{wal}_k(x)} = \omega_b^{-\sum_{i\geq 0} \kappa_i \xi_{i+1}} = \omega_b^{\sum_{i\geq 0} (\ominus \kappa_i) \xi_{i+1}} = {}_b\mathrm{wal}_{\ominus k}(x). \qquad \square$$

As corollary to Proposition A.4 we get its multi-dimensional analogue.

Corollary A.5 *For all $\boldsymbol{k}, \boldsymbol{l} \in \mathbb{N}_0^s$ we have*

$$ {}_b\mathrm{wal}_{\boldsymbol{k}} \cdot {}_b\mathrm{wal}_{\boldsymbol{l}} = {}_b\mathrm{wal}_{\boldsymbol{k} \oplus \boldsymbol{l}} \quad \textit{and} \quad \frac{1}{{}_b\mathrm{wal}_{\boldsymbol{k}}} = \overline{{}_b\mathrm{wal}_{\boldsymbol{k}}} = {}_b\mathrm{wal}_{\ominus \boldsymbol{k}}.$$

Proposition A.6 *Let $k \in \mathbb{N}_0$, then for all $x, y \in [0,1)$ for which $x \oplus y$ and $x \ominus y$ respectively is defined we have*

$$ {}_b\mathrm{wal}_k(x)\, {}_b\mathrm{wal}_k(y) = {}_b\mathrm{wal}_k(x \oplus y) \ \textit{and} \ {}_b\mathrm{wal}_k(x) \overline{{}_b\mathrm{wal}_k(y)} = {}_b\mathrm{wal}_k(x \ominus y),$$

respectively.

Proof Let $k = \kappa_0 + \kappa_1 b + \kappa_2 b^2 + \cdots$ and assume that $x = \xi_1 b^{-1} + \xi_2 b^{-2} + \cdots$ and $y = \eta_1 b^{-1} + \eta_2 b^{-2} + \cdots$ satisfy the condition from the statement of the proposition. Then we have

$$\begin{aligned} {}_b\mathrm{wal}_k(x)\, {}_b\mathrm{wal}_k(y) &= \omega_b^{\sum_{i\geq 0} \kappa_i \xi_{i+1}} \omega_b^{\sum_{i\geq 0} \kappa_i \eta_{i+1}} \\ &= \omega_b^{\sum_{i\geq 0} \kappa_i (\xi_{i+1} + \eta_{i+1})} = {}_b\mathrm{wal}_k(x \oplus y), \end{aligned}$$

and

$$\begin{aligned} {}_b\mathrm{wal}_k(x)\,\overline{{}_b\mathrm{wal}_k(y)} &= \omega_b^{\sum_{i\geq 0}\kappa_i\xi_{i+1}}\,\omega_b^{-\sum_{i\geq 0}\kappa_i\eta_{i+1}} \\ &= \omega_b^{\sum_{i\geq 0}\kappa_i(\xi_{i+1}-\eta_{i+1})} = {}_b\mathrm{wal}_k(x\ominus y). \end{aligned}$$ □

Again, as corollary to Proposition A.6 we get its multi-dimensional analogue.

Corollary A.7 *Let $\boldsymbol{k}\in\mathbb{N}_0^s$, then for all $\boldsymbol{x},\boldsymbol{y}\in[0,1)^s$, for which $\boldsymbol{x}\oplus\boldsymbol{y}$ and $\boldsymbol{x}\ominus\boldsymbol{y}$ respectively is defined we have*

$${}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x})\cdot{}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{y}) = {}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}\oplus\boldsymbol{y}) \text{ and } {}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x})\cdot\overline{{}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{y})} = {}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}\ominus\boldsymbol{y}),$$

respectively.

Lemma A.8 *For $1\leq k<b^r$ we have $\sum_{a=0}^{b^r-1}{}_b\mathrm{wal}_k(a/b^r)=0$.*

Proof Let $k=\kappa_0+\kappa_1 b+\cdots+\kappa_{r-1}b^{r-1}$. For $\kappa\in\{1,\ldots,b-1\}$ we have $\sum_{a=0}^{b-1}\omega_b^{\kappa a}=0$ by the formula for a geometric sum and hence

$$\sum_{a=0}^{b^r-1}{}_b\mathrm{wal}_k(a/b^r) = \sum_{a_0,\ldots,a_{r-1}=0}^{b-1}\omega_b^{\kappa_0 a_{r-1}+\cdots+\kappa_{r-1}a_0} = \prod_{i=0}^{r-1}\sum_{a=0}^{b-1}\omega_b^{\kappa_i a}=0,$$

as there is an $i\in\{0,1,\ldots,r-1\}$ such that $\kappa_i\neq 0$. □

Proposition A.9 *We have*

$$\int_0^1{}_b\mathrm{wal}_k(x)\,\mathrm{d}x = \begin{cases} 1 & \text{if } k=0,\\ 0 & \text{if } k\neq 0.\end{cases}$$

Proof We have ${}_b\mathrm{wal}_0\equiv 1$ and hence the integral is 1 for $k=0$. Let now $k=\kappa_0+\kappa_1 b+\cdots+\kappa_{r-1}b^{r-1}$ with $\kappa_{r-1}\neq 0$. From Proposition A.2 we know that ${}_b\mathrm{wal}_k$ is constant on the elementary intervals of order r. Then we have

$$\int_0^1{}_b\mathrm{wal}_k(x)\,\mathrm{d}x = \sum_{a=0}^{b^r-1}\int_{a/b^r}^{(a+1)/b^r}{}_b\mathrm{wal}_k(x)\,\mathrm{d}x = \frac{1}{b^r}\sum_{a=0}^{b^r-1}{}_b\mathrm{wal}_k(a/b^r)$$

and the result follows from Lemma A.8. □

The next result shows that the s-dimensional Walsh function system is orthonormal in $L_2([0,1]^s)$.

Proposition A.10 *For all $\boldsymbol{k},\boldsymbol{l}\in\mathbb{N}_0^s$ we have*

$$\int_{[0,1]^s}{}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x})\,\overline{{}_b\mathrm{wal}_{\boldsymbol{l}}(\boldsymbol{x})}\,\mathrm{d}\boldsymbol{x} = \begin{cases} 1 & \text{if } \boldsymbol{k}=\boldsymbol{l},\\ 0 & \text{if } \boldsymbol{k}\neq\boldsymbol{l}.\end{cases}$$

Proof By Corollary A.5 we have ${}_b\mathrm{wal}_{\boldsymbol{k}} \cdot \overline{{}_b\mathrm{wal}_{\boldsymbol{l}}} = {}_b\mathrm{wal}_{\boldsymbol{k}\ominus\boldsymbol{l}}$. Hence

$$\int_{[0,1]^s} {}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x})\,\overline{{}_b\mathrm{wal}_{\boldsymbol{l}}(\boldsymbol{x})}\,\mathrm{d}\boldsymbol{x} = \int_{[0,1]^s} {}_b\mathrm{wal}_{\boldsymbol{k}\ominus\boldsymbol{l}}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$$

and the result follows from Proposition A.9. □

Theorem A.11 *For fixed $b, s \in \mathbb{N}$, $b \geq 2$, the s-dimensional b-adic Walsh function system is a complete orthonormal basis in $L_2([0,1]^s)$.*

For the proof of this fundamental result we need some preparation.

Lemma A.12 *Let $b \geq 2$ be an integer. Then the one-dimensional Lebesgue measure λ is invariant under digit wise addition modulo b. In other words, for all $M \subseteq [0,1)$ which is Lebesgue measurable and for all $x \in [0,1)$ we have $\lambda(M) = \lambda(M \oplus x)$, where $M \oplus x := \{y \oplus x : y \in M\}$.*

Proof Let $x \in [0,1)$ and $y \in M$ with $x = \xi_1 b^{-1} + \xi_2 b^{-2} + \cdots$ and $y = \eta_1 b^{-1} + \eta_2 b^{-2} + \cdots$. Then $x \oplus y$ is not defined, if $\xi_j + \eta_j \equiv b-1 \pmod b$ or equivalently $\eta_j \equiv b-1-\xi_j \pmod b$ for all indices $j \geq j_0$. Hence, the subset $\{y \in M : y \oplus x \text{ not defined}\}$ is countable.

Consider an elementary interval $J = [a/b^r, (a+1)/b^r)$ with $a = \alpha_0 + \alpha_1 b + \cdots + \alpha_{r-1} b^{r-1}$. Each $y \in I$ has the b-adic expansion $y = \alpha_{r-1} b^{-1} + \alpha_{r-2} b^{-2} + \cdots + \alpha_0 b^{-r} + \eta_{r+1} b^{-(r+1)} + \eta_{r+2} b^{-(r+2)} + \cdots$ with digits $0 \leq \eta_j \leq b-1$ for all $j \geq r+1$.

Now for $y \in J$ we have

$$y \oplus x = \frac{\alpha_{r-1} \oplus \xi_1}{b} + \cdots + \frac{\alpha_0 \oplus \xi_r}{b^r} + \frac{\eta_{r+1} \oplus \xi_{r+1}}{b^{r+1}} + \frac{\eta_{r+2} \oplus \xi_{r+2}}{b^{r+2}} + \cdots .$$

Hence, $y \mapsto y \oplus x$ maps all but countably many points from J to the elementary interval

$$J' = \left[\frac{\alpha_{r-1} \oplus x_1}{b} + \cdots + \frac{\alpha_0 \oplus x_r}{b^r}, \frac{\alpha_{r-1} \oplus x_1}{b} + \cdots + \frac{\alpha_0 \oplus x_r}{b^r} + \frac{1}{b^r} \right).$$

Furthermore, for all but countably many points $y \in J'$ we can define the inverse mapping $y \mapsto y \ominus x$. Hence $\oplus x$ preserves the measure of elementary intervals.

Since every open subset from $[0,1]$ can be written as a countable union of elementary intervals it follows that $y \mapsto y \oplus x$ preserves the measure of every open subset of $[0,1)$ and hence the result follows for all Lebesgue measurable subsets from $[0,1)$. □

Corollary A.13 *Let $\boldsymbol{c} \in [0,1)^s$, then for all $f \in L_2([0,1]^s)$ we have*

$$\int_{[0,1]^s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_{[0,1]^s} f(\boldsymbol{x} \oplus \boldsymbol{c})\,\mathrm{d}\boldsymbol{x}.$$

Proof It is enough to show the result for $s = 1$. Let $c \in [0,1)$ and let $f \in L_2([0,1])$. Define $g(x) = f(x \oplus c)$. For each $M \subseteq f([0,1])$ we have $g^{-1}(M) = f^{-1}(M) \ominus c$ and hence, by Lemma A.12, we have $\lambda(g^{-1}(M)) = \lambda(f^{-1}(M))$. Now the result follows from the definition of the Lebesgue-integral. □

Definition A.14 An *s-dimensional b-adic Walsh series* is a function $f : [0,1]^s \to \mathbb{C}$ of the form

$$f = \sum_{\boldsymbol{k} \in \mathbb{N}_0^s} \widehat{f}(\boldsymbol{k})\, {}_b\mathrm{wal}_{\boldsymbol{k}}$$

for certain $\widehat{f}(\boldsymbol{k}) \in \mathbb{C}$ which are called the *Walsh coefficients* or *Walsh-Fourier coefficients* of the function f. Furthermore, a *Walsh polynomial* is a finite Walsh series.

Remark A.15 For uniformly convergent Walsh series f one can compute the $\boldsymbol{k}$th Walsh coefficient by

$$\widehat{f}(\boldsymbol{k}) = \int_{[0,1]^s} f(\boldsymbol{x})\, \overline{{}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x})}\,\mathrm{d}\boldsymbol{x}.$$

We introduce very special Walsh polynomials, the so-called Walsh-Dirichlet kernels.

Definition A.16 For $\boldsymbol{k} = (k_1, \ldots, k_s) \in \mathbb{N}^s$ the *$\boldsymbol{k}$th Walsh-Dirichlet kernel* is defined as

$$D_{\boldsymbol{k}} = \sum_{l_1=0}^{k_1-1} \cdots \sum_{l_s=0}^{k_s-1} {}_b\mathrm{wal}_{l_1,\ldots,l_s}.$$

Lemma A.17 *For $n \in \mathbb{N}_0$ let $\boldsymbol{b}_n = (b^n, \ldots, b^n) \in \mathbb{N}^s$, then we have $D_{\boldsymbol{b}_n}(x) = b^{ns}\chi_{[0,b^{-n})^s}(x)$ for $\boldsymbol{x} \in [0,1)^s$.*

Proof For $s = 1$ we show the result by induction on $n \in \mathbb{N}_0$. Let $x \in [0,1)$.

We have $D_1(x) = {}_b\mathrm{wal}_0(x) = 1 = b^{0\cdot s}\chi_{[0,b^{-0})}(x)$ and hence the result holds for $n = 0$. Assume the formula holds for $D_{b^{n-1}}(x)$. Then we have

$$D_{b^n}(x) = \sum_{l=0}^{b^n-1} {}_b\mathrm{wal}_l(x) = \sum_{i=0}^{b-1} \sum_{l=0}^{b^{n-1}-1} {}_b\mathrm{wal}_{l \oplus i\cdot b^{n-1}}(x)$$

$$= \sum_{i=0}^{b-1} {}_b\mathrm{wal}_{i\cdot b^{n-1}}(x) \sum_{l=0}^{b^{n-1}-1} {}_b\mathrm{wal}_l(x)$$
$$= D_{b^{n-1}}(x) \sum_{i=0}^{b-1} {}_b\mathrm{wal}_{i\cdot b^{n-1}}(x) = D_{b^{n-1}}(x) \sum_{i=0}^{b-1} (\omega_b^{\xi_n})^i,$$

where ξ_n is the nth digit of x in its base b expansion. The last sum is equal to $b^{n-1}b = b^n$ if $x \in [0, b^{-n+1})$ and $\xi_n = 0$ which is equivalent to $x \in [0, b^{-n})$ and equal to 0 in all other cases. Hence the result follows for $s = 1$.

For $s > 1$ the result follows immediately from the identity

$$D_{\boldsymbol{b}_n}(\boldsymbol{x}) = \sum_{l_1,\dots,l_s=0}^{b^n-1} {}_b\mathrm{wal}_{l_1,\dots,l_s}(\boldsymbol{x}) = \prod_{i=1}^{s} \sum_{l_i=0}^{b^n-1} {}_b\mathrm{wal}_{l_i}(x_i) = \prod_{i=1}^{s} D_{b^n}(x_i)$$

together with the result for the case $s = 1$. □

Proof of Theorem A.11 We know already from Proposition A.10 that Walsh functions are orthonormal in $L_2([0,1]^s)$. Hence it remains to show that the Walsh polynomials are dense in $L_2([0,1]^s)$.

For $\boldsymbol{n} = (n_1, \dots, n_s) \in \mathbb{N}^s$ denote by $S_{\boldsymbol{n}}(\boldsymbol{x}, f)$ the $\boldsymbol{n}$th partial sum of the form

$$S_{\boldsymbol{n}}(\boldsymbol{x}, f) = \sum_{l_1=0}^{n_1-1} \cdots \sum_{l_s=0}^{n_s-1} \widehat{f}(\boldsymbol{l})\, {}_b\mathrm{wal}_{l_1,\dots,l_s}(\boldsymbol{x}),$$

with $\widehat{f}(\boldsymbol{l}) = \int_{[0,1]^s} f(\boldsymbol{t})\, \overline{{}_b\mathrm{wal}_{\boldsymbol{l}}(\boldsymbol{t})}\, \mathrm{d}\boldsymbol{t}$. Then we have

$$\begin{aligned} S_{\boldsymbol{n}}(\boldsymbol{x}, f) &= \sum_{l_1=0}^{n_1-1} \cdots \sum_{l_s=0}^{n_s-1} \left(\int_{[0,1]^s} f(\boldsymbol{t})\, \overline{{}_b\mathrm{wal}_{l_1,\dots,l_s}(\boldsymbol{t})}\, \mathrm{d}\boldsymbol{t} \right) {}_b\mathrm{wal}_{l_1,\dots,l_s}(\boldsymbol{x}) \\ &= \int_{[0,1]^s} f(\boldsymbol{t}) \sum_{l_1=0}^{n_1-1} \cdots \sum_{l_s=0}^{n_s-1} {}_b\mathrm{wal}_{l_1,\dots,l_s}(\boldsymbol{x} \ominus \boldsymbol{t})\, \mathrm{d}\boldsymbol{t} \\ &= \int_{[0,1]^s} f(\boldsymbol{t}) D_{\boldsymbol{n}}(\boldsymbol{x} \ominus \boldsymbol{t})\, \mathrm{d}\boldsymbol{t} = \int_{[0,1]^s} f(\boldsymbol{x} \ominus \boldsymbol{t}) D_{\boldsymbol{n}}(\boldsymbol{t})\, \mathrm{d}\boldsymbol{t}. \end{aligned}$$

With the help of this formula we can now estimate the approximation error for certain partial sums. With Lemma A.17 we obtain

$$|S_{\boldsymbol{b}_n}(\boldsymbol{x}, f) - f(\boldsymbol{x})| = \left| \int_{[0,1]^s} f(\boldsymbol{x} \ominus \boldsymbol{t}) D_{\boldsymbol{b}_n}(\boldsymbol{t})\, \mathrm{d}\boldsymbol{t} - b^{ns} \int_{[0,b^{-n})^s} f(\boldsymbol{x})\, \mathrm{d}\boldsymbol{t} \right|$$

$$
\begin{aligned}
&= b^{ns} \left| \int_{[0,b^{-n})^s} (f(\boldsymbol{x} \ominus \boldsymbol{t}) - f(\boldsymbol{x}))\, \mathrm{d}\boldsymbol{t} \right| \\
&= b^{ns} \left| \int_{\boldsymbol{x} \ominus [0,b^{-n})^s} (f(\boldsymbol{t}) - f(\boldsymbol{x}))\, \mathrm{d}\boldsymbol{t} \right| \\
&\leq \sup \left\{ |f(\boldsymbol{t}) - f(\boldsymbol{x})| \,:\, \boldsymbol{x}, \boldsymbol{t} \in \boldsymbol{x} \ominus [0, b^{-n})^s \right\} b^{ns} \lambda_s(\boldsymbol{x} \ominus [0, b^{-n})^s).
\end{aligned}
$$

Assume now that $f \in C([0,1]^s)$. Hence f is also uniformly continuous on $[0,1]^s$ and thus for every $\varepsilon > 0$ there exists an $N_0 = N_0(\varepsilon)$ such that for all $n > N_0$ and for all $\boldsymbol{x}, \boldsymbol{y} \in [0,1]^s$ with $|\boldsymbol{x} - \boldsymbol{y}|_\infty < b^{-n}$ we have $|f(\boldsymbol{x}) - f(\boldsymbol{y})| < \varepsilon$.

Obviously $\lambda_s\left(\boldsymbol{x} \ominus [0, b^{-n})^s\right) = b^{-ns}$ and hence we obtain

$$|S_{\boldsymbol{b}_n}(\boldsymbol{x}, f) - f(\boldsymbol{x})| < \varepsilon$$

for all $n > N_0(\varepsilon)$ and this holds independently from $\boldsymbol{x}$ as f is uniformly continuous. Hence

$$\|S_{\boldsymbol{b}_n}(\boldsymbol{x}, \cdot) - f\|_\infty < \varepsilon$$

for all $n > N_0(\varepsilon)$. This means that the Walsh polynomials are dense in $C([0,1]^s)$ with respect to the sup-norm $\|\cdot\|_\infty$ which in turn is dense in $L_2([0,1]^s)$ with respect to the L_2-norm $\|\cdot\|_2$. Hence the Walsh polynomials are dense also in $L_2([0,1]^s)$.

Since $L_2([0,1]^s)$ is a Hilbert space, we have now that the Walsh functions are a complete orthonormal system in $L_2([0,1]^s)$. □

Note that Bessel's inequality

$$\sum_{\boldsymbol{k} \in \mathbb{N}_0^s} |\widehat{f}(\boldsymbol{k})|^2 \leq \int_{[0,1]^s} |f(\boldsymbol{x})|^2 \, \mathrm{d}\boldsymbol{x} \tag{A.1}$$

holds for functions $f \in L_2([0,1]^s)$. Since

$$
\begin{aligned}
0 &\leq \int_{[0,1]^s} |f(\boldsymbol{x}) - S_{\boldsymbol{n}}(\boldsymbol{x}, f)|^2 \, \mathrm{d}\boldsymbol{x} \\
&= \int_{[0,1]^s} |f(\boldsymbol{x})|^2 \, \mathrm{d}\boldsymbol{x} - \int_{[0,1]^s} f(\boldsymbol{x}) \overline{S_{\boldsymbol{n}}(\boldsymbol{x}, f)} \, \mathrm{d}\boldsymbol{x} \\
&\quad - \int_{[0,1]^s} \overline{f(\boldsymbol{x})} S_{\boldsymbol{n}}(\boldsymbol{x}, f) \, \mathrm{d}\boldsymbol{x} + \int_{[0,1]^s} |S_{\boldsymbol{n}}(\boldsymbol{x}, f)|^2 \, \mathrm{d}\boldsymbol{x} \\
&= \int_{[0,1]^s} |f(\boldsymbol{x})|^2 \, \mathrm{d}\boldsymbol{x} - \sum_{l_1=0}^{n_1-1} \cdots \sum_{l_s=0}^{n_s-1} |\widehat{f}(l_1, \ldots, l_s)|^2,
\end{aligned} \tag{A.2}
$$

we have $\sum_{l_1=0}^{n_1-1} \cdots \sum_{l_s=0}^{n_s-1} |\widehat{f}(l_1,\ldots,l_s)|^2 \le \int_{[0,1]^s} |f(\boldsymbol{x})|^2 \,\mathrm{d}\boldsymbol{x}$, and by considering $n_1,\ldots,n_s \to \infty$ the result follows. We also have the following lemma.

Lemma A.18 *Let $\{a_{\boldsymbol{k}} \in \mathbb{C} : \boldsymbol{k} \in \mathbb{N}_0^s\}$ be a set of complex numbers such that $\sum_{\boldsymbol{k}\in\mathbb{N}_0^s} |a_{\boldsymbol{k}}|^2 < \infty$. Then the Walsh series $\sum_{\boldsymbol{k}\in\mathbb{N}_0^s} a_{\boldsymbol{k}}\, {}_b\mathrm{wal}_{\boldsymbol{k}}$ converges in $L_2([0,1]^s)$.*

Proof Since $L_2([0,1]^s)$ is complete, we only need to show that the partial sums $S_{\boldsymbol{n}} = \sum_{l_1=0}^{n_1-1} \cdots \sum_{l_s=0}^{n_s-1} a_{l_1,\ldots,l_s}\, {}_b\mathrm{wal}_{l_1,\ldots,l_s}$, where $\boldsymbol{n} = (n_1,\ldots,n_s) \in \mathbb{N}^s$, form a Cauchy sequence in $L_2([0,1]^s)$.

Indeed, for any $\boldsymbol{n},\boldsymbol{n}' \in \mathbb{N}_0^s$ with $\boldsymbol{n} = (n_1,\ldots,n_s)$ and $\boldsymbol{n}' = (n_1',\ldots,n_s')$, where we assume that $n_1 > n_1',\ldots,n_s > n_s'$, we have

$$\int_{[0,1]^s} |S_{\boldsymbol{n}}(\boldsymbol{x},f) - S_{\boldsymbol{n}'}(\boldsymbol{x},f)|^2 \,\mathrm{d}\boldsymbol{x} = \sum_{l_1=n_1'}^{n_1-1} \cdots \sum_{l_s=n_s'}^{n_s-1} |a_{l_1,\ldots,l_s}|^2 \to 0$$

as $n_1',\ldots,n_s' \to \infty$. Thus the partial sums $S_{\boldsymbol{n}}$ form a Cauchy sequence and hence the result follows as $L_2([0,1]^s)$ is complete. □

The completeness of the Walsh function system shown in Theorem A.11 is equivalent to the statement that Plancherel's identity

$$\int_{[0,1]^s} |f(\boldsymbol{x})|^2 \,\mathrm{d}\boldsymbol{x} = \sum_{\boldsymbol{k}\in\mathbb{N}_0^s} |\widehat{f}(\boldsymbol{k})|^2$$

holds. This is shown in the following theorem (see for example [117, Section I.5] for a more general statement).

Theorem A.19 *The following statements are equivalent:*

(a) The Walsh function system is complete in $L_2([0,1]^s)$.

(b) For every $f \in L_2([0,1]^s)$ we have

$$\int_{[0,1]^s} |f(\boldsymbol{x})|^2 \,\mathrm{d}\boldsymbol{x} = \sum_{\boldsymbol{k}\in\mathbb{N}_0^s} |\widehat{f}(\boldsymbol{k})|^2.$$

(c) For every $f \in L_2([0,1]^s)$ we have

$$\lim_{n_1,\ldots,n_s\to\infty} \int_{[0,1]^s} |f(\boldsymbol{x}) - S_{n_1,\ldots,n_s}(\boldsymbol{x},f)|^2 \,\mathrm{d}\boldsymbol{x} = 0.$$

Proof The equivalence of (*b*) and (*c*) follows from (A.2).

Assume now that (*b*) holds. Let $\langle g,h\rangle_{L_2} = \int_{[0,1]^s} g(\boldsymbol{x})\overline{h(\boldsymbol{x})} \,\mathrm{d}\boldsymbol{x}$ denote the inner product in $L_2([0,1]^s)$. If a function $f \in L_2([0,1]^s)$ is orthogonal to ${}_b\mathrm{wal}_{\boldsymbol{k}}$ for all $\boldsymbol{k} \in \mathbb{N}_0^s$ it follows that $\widehat{f}(\boldsymbol{k}) = \langle f, {}_b\mathrm{wal}_{\boldsymbol{k}}\rangle_{L_2} = 0$ for all $\boldsymbol{k} \in \mathbb{N}_0^s$

and (b) implies that $\int_{[0,1]^s} |f(\boldsymbol{x})|^2 \,\mathrm{d}\boldsymbol{x} = 0$. Thus the Walsh function system is complete in $L_2([0,1]^s)$ and hence (b) implies (a).

Assume now that (a) holds. We complete the proof by showing that (c) follows. From Bessel's inequality and Lemma A.18 it follows that for every $f \in L_2([0,1]^s)$ we have $S(\cdot, f) := \sum_{\boldsymbol{k} \in \mathbb{N}_0^s} \widehat{f}(\boldsymbol{k})\, {}_b\mathrm{wal}_{\boldsymbol{k}} \in L_2([0,1]^s)$. Hence $\langle f - S(\cdot, f),\, {}_b\mathrm{wal}_{\boldsymbol{k}} \rangle = 0$ for all $\boldsymbol{k} \in \mathbb{N}_0^s$. Thus, if the Walsh function system $\{ {}_b\mathrm{wal}_{\boldsymbol{k}} : \boldsymbol{k} \in \mathbb{N}_0^s \}$ is complete, it follows that $\int_{[0,1]^s} |f(\boldsymbol{x}) - S(\boldsymbol{x}, f)|^2 \,\mathrm{d}\boldsymbol{x} = 0$. □

A.3 Convergence of the Walsh series

For our purposes here we need strong assumptions on the convergence of the Walsh series $\sum_{k=0}^{\infty} \widehat{f}(k)\, {}_b\mathrm{wal}_k(x)$ to the function f, i.e., we require that the partial series $\sum_{k=0}^{L} \widehat{f}(k)\, {}_b\mathrm{wal}_k(x)$ converges to $f(x)$ at every point $x \in [0,1)$ as $L \to \infty$.

For continuous functions $f : [0,1) \to \mathbb{R}$ we can use the argument in [77, p. 373] to show that certain partial sums of the Walsh series converge at every point $x \in [0,1)$ to the function value $f(x)$. Indeed, for a given $x \in [0,1)$ we have

$$\begin{aligned} \sum_{k=0}^{b^\ell - 1} \widehat{f}(k)\, {}_b\mathrm{wal}_k(x) &= \int_0^1 f(y) \sum_{k=0}^{b^\ell - 1} {}_b\mathrm{wal}_k(x)\, \overline{{}_b\mathrm{wal}_k(y)} \,\mathrm{d}y \\ &= b^\ell \int_{b^{-\ell}\lfloor b^\ell x \rfloor}^{b^{-\ell}\lfloor b^\ell x \rfloor + b^{-\ell}} f(y) \,\mathrm{d}y. \end{aligned}$$

As the function f is continuous it follows that $\sum_{k=0}^{b^l - 1} \widehat{f}(k)\, {}_b\mathrm{wal}_k(x)$ converges to $f(x)$ as $l \to \infty$. Hence, if the partial sums $\sum_{k=0}^{L} \widehat{f}(k)\, {}_b\mathrm{wal}_k(x)$ are a Cauchy sequence, then we also have that $\sum_{k=0}^{L} \widehat{f}(k)\, {}_b\mathrm{wal}_k(x)$ converges to $f(x)$ as $L \to \infty$.

For instance, if $\sum_{k=0}^{\infty} |\widehat{f}(k)| < \infty$, then the partial sums $\sum_{k=0}^{L} \widehat{f}(k)\, {}_b\mathrm{wal}_k(x)$ are a Cauchy sequence and hence $\sum_{k=0}^{b^l - 1} \widehat{f}(k)\, {}_b\mathrm{wal}_k(x)$ converges to $f(x)$ as $l \to \infty$. In this case the convergence is even uniformly in x.

We have shown the following result which is sufficient for our purposes. For more elaborate results in this direction see [230].

Theorem A.20 *Let $f : [0,1] \to \mathbb{R}$ be a continuous function and assume that $\sum_{k=0}^{\infty} |\widehat{f}(k)| < \infty$. Then $\sum_{k=0}^{L} \widehat{f}(k)\, {}_b\mathrm{wal}_k(x)$ converges uniformly to*

$f(x)$ as $L \to \infty$ and we have

$$f(x) = \sum_{k=0}^{\infty} \widehat{f}(k)\, {}_b\mathrm{wal}_k(x) \quad \textit{for all } x \in [0,1).$$

Remark A.21 We remark that in [258] it was shown that there are continuous functions f for which $\sum_{k=0}^{L} \widehat{f}(k)\, {}_b\mathrm{wal}_k(x)$ does not converge at some given point x as $L \to \infty$. Therefore continuity is not a sufficient condition to ensure that $\sum_{k=0}^{L} \widehat{f}(k)\, {}_b\mathrm{wal}_k(x)$ is a Cauchy sequence.

However, Walsh [258] already proved the following result. If the function f is not merely continuous, but has bounded variation (for instance, if f has a derivative which is square integrable, i.e., $\int_0^1 |f'(y)|^2 \,\mathrm{d}y < \infty$, then f has bounded variation), then it follows that $\sum_{k=0}^{L} \widehat{f}(k)\, {}_b\mathrm{wal}_k(x)$ is a Cauchy sequence for every x. Hence, in this case we have $\sum_{k=0}^{L} {}_b\mathrm{wal}_k(x) \to f(x)$ as $L \to \infty$ for all x.

The argument above can also be extended to continuous functions $f : [0,1)^s \to \mathbb{R}$, see Exercise A.9.

A.4 Walsh series expansions of a certain function

In this section we provide the b-adic Walsh series representations of a functions which is used throughout this book.

Lemma A.22 *For $b \geq 2$ an integer and $x \in [0,1)$ we have*

$$x - \frac{1}{2} = \sum_{a=1}^{\infty} \sum_{\kappa=1}^{b-1} \frac{1}{b^a(\omega_b^{-\kappa} - 1)}\, {}_b\mathrm{wal}_{\kappa b^{a-1}}(x). \tag{A.3}$$

Proof Let $x = \xi_1 b^{-1} + \xi_2 b^{-2} + \cdots$ and $k = \kappa_{a-1} b^{a-1} + \cdots + \kappa_1 b + \kappa_0$, where $\kappa_{a-1} \neq 0$. Then we have

$$\begin{aligned}
&\int_0^1 \left(x - \tfrac{1}{2}\right) \overline{{}_b\mathrm{wal}_k(x)}\,\mathrm{d}x \\
&= \sum_{\xi_1=0}^{b-1} \cdots \sum_{\xi_a=0}^{b-1} \omega_b^{-(\xi_1\kappa_0 + \cdots + \xi_a\kappa_{a-1})} \int_{\frac{\xi_1}{b} + \cdots + \frac{\xi_a}{b^a}}^{\frac{\xi_1}{b} + \cdots + \frac{\xi_a}{b^a} + \frac{1}{b^a}} \left(x - \tfrac{1}{2}\right)\,\mathrm{d}x \\
&= \frac{1}{b^a} \sum_{\xi_1=0}^{b-1} \omega_b^{-\xi_1\kappa_0} \cdots \sum_{\xi_a=0}^{b-1} \omega_b^{-\xi_a\kappa_{a-1}} \left(\frac{\xi_1}{b} + \cdots + \frac{\xi_a}{b^a}\right),
\end{aligned} \tag{A.4}$$

where we used the facts that

$$\int_{\frac{\xi_1}{b}+\cdots+\frac{\xi_a}{b^a}}^{\frac{\xi_1}{b}+\cdots+\frac{\xi_a}{b^a}+\frac{1}{b^a}} \left(x-\tfrac{1}{2}\right)\,\mathrm{d}x = \frac{1}{b^a}\left(\frac{\xi_1}{b}+\cdots+\frac{\xi_a}{b^a}\right)+\frac{1}{2\cdot b^a}\left(\frac{1}{b^a}-1\right)$$

and $\sum_{\xi_a=0}^{b-1}\omega_b^{-\xi_a\kappa_{a-1}}=0$ for $\kappa_{a-1}\neq 0$. For any digits $0\le\xi_1,\ldots,\xi_{a-1}\le b-1$ we have

$$\begin{aligned}\sum_{\xi_a=0}^{b-1}\left(\frac{\xi_1}{b}+\cdots+\frac{\xi_{a-1}}{b^{a-1}}+\frac{\xi_a}{b^a}\right)\omega_b^{-\xi_a\kappa_{a-1}} &= \sum_{\xi_a=0}^{b-1}\frac{\xi_a}{b^a}\omega_b^{-\xi_a\kappa_{a-1}}\\ &= \frac{b}{b^a(\omega_b^{-\kappa_{a-1}}-1)},\end{aligned}$$

as for $\kappa_{a-1}\neq 0$ we have

$$\sum_{\xi_a=0}^{b-1}\omega_b^{-\xi_a\kappa_{a-1}}=0 \text{ and } \sum_{\xi_a=0}^{b-1}\xi_a\omega_b^{-\xi_a\kappa_{a-1}}=\frac{b}{\omega_b^{-\kappa_{a-1}}-1}. \tag{A.5}$$

Therefore we obtain from (A.4)

$$\int_0^1\left(x-\tfrac{1}{2}\right)\overline{{}_b\mathrm{wal}_k(x)}\,\mathrm{d}x=\frac{b}{b^{2a}(\omega_b^{-\kappa_{a-1}}-1)}\sum_{\xi_1=0}^{b-1}\omega_b^{-\xi_1\kappa_0}\cdots\sum_{\xi_{a-1}=0}^{b-1}\omega_b^{-\xi_{a-1}\kappa_{a-2}}.$$

For an integer $0\le\kappa\le b-1$ we use

$$\sum_{\xi=0}^{b-1}\omega_b^{-\xi\kappa}=\begin{cases} b & \text{if } \kappa=0,\\ 0 & \text{if } \kappa\neq 0,\end{cases}$$

and obtain

$$\int_0^1\left(x-\tfrac{1}{2}\right)\overline{{}_b\mathrm{wal}_k(x)}\,\mathrm{d}x=\begin{cases}\frac{1}{b^a(\omega_b^{-\kappa_{a-1}}-1)} & \text{if } \kappa_0=\cdots=\kappa_{a-2}=0,\\ 0 & \text{otherwise.}\end{cases}$$

Thus, for $x\in[0,1)$, we have

$$x-\frac{1}{2}=\sum_{a=1}^{\infty}\sum_{\kappa=1}^{b-1}\frac{1}{b^a(\omega_b^{-\kappa}-1)}\,{}_b\mathrm{wal}_{\kappa b^{a-1}}(x). \qquad \square$$

With Lemma A.22 we can prove a formula for a trigonometric sum which is often used throughout this book.

Corollary A.23 *For $b\ge 2$ and for $l\in\{-(b-1),\ldots,b-1\}$ we have*

$$\sum_{\kappa=1}^{b-1}\frac{\omega_b^{\kappa l}}{\sin^2(\kappa\pi/b)}=2(|l|(|l|-b)+\frac{b^2-1}{3}.$$

In particular, $\sum_{\kappa=1}^{b-1} \sin^{-2}(\kappa\pi/b) = (b^2-1)/3$.

Proof Using Lemma A.22 and the orthogonality properties of the Walsh functions (see Proposition A.10) we obtain (see also (12.8))

$$2\sum_{a=1}^{\infty}\sum_{\kappa=1}^{b-1}\frac{1}{b^{2a}|\omega_b^\kappa - 1|^2} = \int_0^1\int_0^1 |x-y|^2\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{6}$$

and hence

$$\sum_{\kappa=1}^{b-1}\frac{1}{|\omega_b^\kappa - 1|^2} = \frac{b^2-1}{12}.$$

For $1 \le l \le b-1$ we use (12.7), which states that for any $x, y \in [0,1)$ we have

$$\begin{aligned}&\int_0^1 |(x\oplus\sigma) - (y\oplus\sigma)|^2\,\mathrm{d}\sigma\\ &= \frac{1}{6} - 2\sum_{a=1}^{\infty}\sum_{\kappa=1}^{b-1}\frac{1}{b^{2a}|\omega_b^\kappa-1|^2}\,{}_b\mathrm{wal}_{\kappa b^{a-1}}(x)\,\overline{{}_b\mathrm{wal}_{\kappa b^{a-1}}(y)}.\end{aligned}$$

Take $x = l/b$ and $y = 0$, then the left hand side of the equation yields

$$\begin{aligned}&\int_0^1 |(x\oplus\sigma) - (y\oplus\sigma)|^2\,\mathrm{d}\sigma = \int_0^1 \left|\left(\frac{l}{b}\oplus\sigma\right) - \sigma\right|^2\,\mathrm{d}\sigma\\ &= \int_0^{\frac{b-l}{b}} \left|\left(\frac{l}{b}+\sigma\right) - \sigma\right|^2\,\mathrm{d}\sigma + \int_{\frac{b-l}{b}}^1 \left|\left(\frac{l-b}{b}+\sigma\right) - \sigma\right|^2\,\mathrm{d}\sigma\\ &= \int_0^{\frac{b-l}{b}} \frac{l^2}{b^2}\,\mathrm{d}\sigma + \int_{\frac{b-l}{b}}^1 \frac{(l-b)^2}{b^2}\,\mathrm{d}\sigma\\ &= \frac{b-l}{b}\frac{l^2}{b^2} + \left(1-\frac{b-l}{b}\right)\frac{(l-b)^2}{b^2}\\ &= \frac{l(b-l)}{b^2},\end{aligned}$$

and for the right hand side we obtain

$$\begin{aligned}&\frac{1}{6} - 2\sum_{a=1}^{\infty}\sum_{\kappa=1}^{b-1}\frac{1}{b^{2a}|\omega_b^\kappa-1|^2}\,{}_b\mathrm{wal}_{\kappa b^{a-1}}(x)\,\overline{{}_b\mathrm{wal}_{\kappa b^{a-1}}(y)}\\ &= \frac{1}{6} - 2\sum_{\kappa=1}^{b-1}\frac{\omega_b^{l\kappa}}{b^2|\omega_b^\kappa-1|^2} - 2\sum_{a=2}^{\infty}\frac{1}{b^{2a}}\sum_{\kappa=1}^{b-1}\frac{1}{|\omega_b^\kappa-1|^2}\end{aligned}$$

$$= \frac{b^2-1}{6b^2} - \frac{2}{b^2}\sum_{\kappa=1}^{b-1}\frac{\omega_b^{l\kappa}}{|\omega_b^\kappa - 1|^2}.$$

Thus, for $0 \le l \le b-1$ we have

$$\sum_{\kappa=1}^{b-1}\frac{\omega_b^{\kappa l}}{|\omega_b^\kappa - 1|^2} = \frac{|l|(|l|-b)}{2} + \frac{b^2-1}{12}. \tag{A.6}$$

To show that (A.6) holds for $-(b-1) \le l \le -1$, use $x = 0$ and $y = -l/b$ in the argument above. The details are omitted.

Further observe that $|\omega_b^\kappa - 1|^2 = |\mathrm{e}^{\pi \mathrm{i}\kappa/b}|^2 |\mathrm{e}^{\pi\mathrm{i}\kappa/b} - \mathrm{e}^{-\pi\mathrm{i}\kappa/b}|^2 = 4\sin^2(\kappa\pi/b)$ and therefore we have the desired result for $-(b-1) \le l \le b-1$. □

Exercises

A.1 Show that a Walsh function can only take finitely many function values, namely the bth roots of unity.

A.2 For $k \in \mathbb{N}_0$, the *kth Rademacher function* $r_k : \mathbb{R} \to \mathbb{R}$, periodic with period one, is defined by $r_0(x) = 1$ for all $x \in [0,1)$ and for $k \in \mathbb{N}$, $r_k(x) = (-1)^j$ if $x \in [j/2^k, (j+1)/2^k)$ for some integer $0 \le j \le 2^k - 1$.

1. Show that the system of Rademacher functions is a sub-class of the dyadic (i.e., $b = 2$) Walsh function system.
2. For $b = 2$, give a definition of Walsh functions in terms of Rademacher functions.

A.3 Let $\boldsymbol{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ with $b^{r_i} \le k_i < b^{r_i+1}$ for all $1 \le i \le s$. Show that ${}_b\mathrm{wal}_{\boldsymbol{k}}$ is constant on an elementary interval of the form

$$\prod_{i=1}^{s}\left[\frac{a_i}{b^{r_i}}, \frac{a_i+1}{b^{r_i}}\right)$$

where $0 \le a_i < b^{r_i}$ are integers for all $1 \le i \le s$.

A.4 Show that we have

$$\frac{1}{{}_b\mathrm{wal}_k(x)} = \overline{{}_b\mathrm{wal}_k(x)} = {}_b\mathrm{wal}_k(\ominus x)$$

whenever $\ominus x$ is defined.

A.5 Let $f : [0,1]^s \to \mathbb{R}$ be a function which is constant on any interval of the form $\prod_{i=1}^s [a_i b^{-n}, (a_i+1)b^{-n})$ with integers $0 \le a_i < b^n$. Show that f is a Walsh polynomial.

A.6 Show that for all $\boldsymbol{k} \in \mathbb{N}^s$ we have $\int_{[0,1]^s} D_{\boldsymbol{k}}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 1$.

A.7 Verify Bessel's inequality (A.1) for the function $f(x) = x^2$.

A.8 Show that the Rademacher functions, defined in Exercise A.2, are not complete.

A.9 Show that the result on the convergence of Walsh series in Section A.3 also holds for s-dimensional continuous functions $f : [0,1]^s \to \mathbb{R}$ for which $\sum_{\boldsymbol{k} \in \mathbb{N}_0^s} |\widehat{f}(\boldsymbol{k})| < \infty$. *Hint:* See [37, Section 3.3] for the result.

A.10 Define a function $f : [0,1) \to \mathbb{R}$ for which $\sum_{k=0}^{\infty} |\widehat{f}(k)| = \infty$.

A.11 Let $b = 2$ and $f(x) = x$. Draw the graphs of f and of $\sum_{k=0}^{2^l-1} \widehat{f}(k)\, {}_b\mathrm{wal}_k(x)$ for $l = 0, 1, 2, 3$.

Notation

Note: In the following we list only symbols that are used in a global context.

Some specific sets and numbers

$\mathbb{N}$	Positive Integers.
$\mathbb{N}_0$	Nonnegative Integers.
$\mathbb{Z}$	Integers.
$\mathbb{R}$	Real numbers.
$\mathbb{C}$	Complex numbers.
$\mathbb{Z}_b$	Residue class ring modulo b (we identify $\mathbb{Z}_b$ with $\{0,\ldots,b-1\}$ with addition and multiplication modulo b).
$\mathbb{F}_b$	Finite field with b elements for a prime power b (if b is a prime, then we identify $\mathbb{F}_b$ with $\mathbb{Z}_b$). The elements of $\mathbb{F}_b$ (for b not a prime) are sometimes denoted by $\overline{0},\overline{1},\ldots,\overline{b-1}$.
$\lvert X\rvert$	Cardinality of a set X.
X^m	The m fold Cartesian product of a set X.
$(X^m)^\top$	The set of m-dimensional column vectors over X.
$\mathcal{P}$	Finite point set in $[0,1)^s$ (interpreted in the sense of the combinatorial notion of "multiset", i.e., a set in which the multiplicity of elements matters.
$\mathcal{S}$	Infinite sequence in $[0,1)^s$.
$\mathcal{I}_s$	Index set $\{1,\ldots,s\}$.
$\mathfrak{u},\mathfrak{v},\ldots$	Subsets of $\mathcal{I}_s$.
$\mathcal{P}_\mathfrak{u}$	Point set in $[0,1)^{\lvert\mathfrak{u}\rvert}$ consisting of the points from $\mathcal{P}$ projected to the components given by $\mathfrak{u}\subseteq\mathcal{I}_s$.
$\mathbb{F}_b[x]$, $\mathbb{Z}_b[x]$	Set of polynomials over $\mathbb{F}_b$ or $\mathbb{Z}_b$.

$\mathbb{F}_b((x^{-1}))$, $\mathbb{Z}_b((x^{-1}))$	Field of formal Laurent series over $\mathbb{F}_b$ or $\mathbb{Z}_b$.
$G_{b,m}$	$G_{b,m} = \{q \in \mathbb{F}_b[x] : \deg(q) < m\}$.
$\boldsymbol{\gamma}$	Set of nonnegative weights, i.e., $\boldsymbol{\gamma} = \{\gamma_{\mathfrak{u}} : \mathfrak{u} \subseteq \mathcal{I}_s\}$. In the case of product weights $\boldsymbol{\gamma} = (\gamma_i)_{i \ge 1}$ is understood as the sequence of one-dimensional weights. In this case we set $\gamma_{\mathfrak{u}} = \prod_{i \in \mathfrak{u}} \gamma_i$.
$\mathtt{i}$	$\mathtt{i} = \sqrt{-1}$.
ω_b	$\omega_b = \mathrm{e}^{2\pi \mathtt{i}}$.

Vectors and matrices

$\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \ldots, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$	Row vectors over $\mathbb{N}, \mathbb{N}_0, \mathbb{Z}$ or $\mathbb{R}$.
$\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots, \mathbf{x}, \mathbf{y}, \mathbf{z}$	Row vectors over $\mathbb{F}_b$ or $\mathbb{Z}_b$.
$\mathbf{a}^\top, \mathbf{b}^\top, \ldots$	Transpose of a vector $\mathbf{a}, \mathbf{b}, \ldots$ in $\mathbb{F}_b$ or $\mathbb{Z}_b$.
$\boldsymbol{x} \cdot \boldsymbol{y}$ (or $\mathbf{x} \cdot \mathbf{y}$)	Usual inner product of the two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ (or $\mathbf{x}$ and $\mathbf{y}$ respectively).
$\boldsymbol{x}_{\mathfrak{u}}$	For an s-dimensional vector $\boldsymbol{x} = (x_1, \ldots, x_s)$ and for $\mathfrak{u} \subseteq \mathcal{I}_s$ the $\lvert\mathfrak{u}\rvert$-dimensional vector consisting of the components of $\boldsymbol{x}$ whose index belongs to $\mathfrak{u}$, i.e., $\boldsymbol{x}_{\mathfrak{u}} = (x_i)_{i \in \mathfrak{u}}$. For example, for $\boldsymbol{x} = (\frac{1}{10}, \frac{1}{3}, \frac{1}{5}, \frac{1}{4}, \frac{1}{8}) \in [0,1)^5$ and $\mathfrak{u} = \{2,3,5\}$ we have $\boldsymbol{x}_{\mathfrak{u}} = (\frac{1}{3}, \frac{1}{5}, \frac{1}{8})$.
$(\boldsymbol{x}_{\mathfrak{u}}, \mathbf{1})$	For an s-dimensional vector $\boldsymbol{x} = (x_1, \ldots, x_s)$ and for $\mathfrak{u} \subseteq \mathcal{I}_s$ the s-dimensional vector whose ith component is x_i if $i \in \mathfrak{u}$ and 1
	$\boldsymbol{x}$ and $\mathfrak{u}$ as above we have $(\boldsymbol{x}_{\mathfrak{u}}, \mathbf{1}) = (1, \frac{1}{3}, \frac{1}{5}, 1, \frac{1}{8})$.
$(\boldsymbol{x}_{\mathfrak{u}}, \mathbf{0})$	Like $(\boldsymbol{x}_{\mathfrak{u}}, \mathbf{1})$ with one replaced by zero.
$(\boldsymbol{x}_{\mathfrak{u}}, \boldsymbol{w})$	For $\boldsymbol{w} = (w_1, \ldots, w_s)$ the vector whose ith component is x_i if $i \in \mathfrak{u}$ and w_i i
$A, B, C, D, \ldots$	$m \times m$ or $\mathbb{N} \times \mathbb{N}$ matrices over $\mathbb{F}_b$.
$A^\top$	Transpose of the matrix A.
$C^{(m)}$	Left upper $m \times m$ sub-matrix of a matrix C.
$C^{(m \times n)}$	Left upper $m \times n$ sub-matrix of a matrix C.

Some specific functions

$d \mid n$, $d \nmid n$	d divides n (d does not divide n).
$\{x\}$	Fractional part of a real number x.
$\lfloor x \rfloor$	Integer part of a real number x, i.e., $\lfloor x \rfloor = x - \{x\}$.
$\lceil x \rceil$	The smallest integer larger than or equal to x.
$\log x$	Natural logarithm of x.
$\log_b x$	Base b logarithm of x.
$\overline{a}$	Complex conjugate of a complex number a.

${}_b\mathrm{wal}_k$	kth b-adic Walsh function (see Definition A.1).
$\chi_J(x)$	Characteristic function of a set J, i.e., $\chi_J(x) = 1$ if $x \in J$ and $\chi_J(x) = 0$ if $x \notin J$.
φ_b	b-adic radical inverse function (see Definition 3.10).
φ	Bijection from $\{0, \dots, b-1\} \to \mathbb{F}_b$.
φ^{-1}	Inverse of the bijection $\varphi : \{0, \dots, b-1\} \to \mathbb{F}_b$.
$A(J, N, \mathcal{S})$	For $\mathcal{S} = (\boldsymbol{x}_n)_{n\geq 0}$ the number of indices n, $0 \leq n < N$, for which the point $\boldsymbol{x}_n$ belongs to J.
$A(J, N, \mathcal{P})$	For a $\mathcal{P} = \{\boldsymbol{x}_0, \dots, \boldsymbol{x}_{N-1}\}$ the number of indices n, $0 \leq n < N$, for which the point $\boldsymbol{x}_n$ belongs to J.
λ_s	s-dimensional Lebesgue measure (for $s = 1$ simply λ).
D_N^*	Star discrepancy (see Definition 2.2 and 2.14).
$D_{N,\boldsymbol{\gamma}}^*$	Weighted star discrepancy (see Definition 3.59).
D_N	Extreme discrepancy (see Definition 3.13).
$L_{q,N}$	L_q-discrepancy (see Definition 3.19).
$L_{q,N,\boldsymbol{\gamma}}$	Weighted L_q-discrepancy (see Definition 3.59).
B_k	kth Bernoulli polynomial.
$O(f(x))$	For $f, g : \mathbb{R} \to \mathbb{R}$, $f \geq 0$, $g(x) = O(f(x))$ for $x \to a$ if there exist $C, \delta > 0$ such that $\lvert g(x)\rvert \leq C f(x)$ for all x with $\lvert x - a\rvert < \delta$ (or $x > \delta$ if $a = \infty$).
$\pi_m(\mathbf{c})$	Projection of $\mathbf{c} \in \mathbb{F}_b^{\mathbb{N}}$ onto its first m components.
$\mathrm{tr}_m(k)$	$\mathrm{tr}_m(k) = \kappa_0 + \kappa_1 b + \cdots + \kappa_{m-1} b^{m-1}$ for $k \in \mathbb{N}_0$ with b-adic expansion $k = \sum_{j\geq 0} \kappa_j b^j$.
$\mathrm{tr}_m(\mathbf{k})$	$\mathrm{tr}_m(\mathbf{k}) = (\kappa_0, \dots, \kappa_{m-1})^\top$ for $k \in \mathbb{N}_0$ with b-adic expansion $k = \sum_{j\geq 0} \kappa_j b^j$.
$I(f)$	Integral of the function f over the s-dimensional unit cube, i.e., $I(f) = \int_{[0,1]^s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$.
$Q_N(f)$	Quasi-Monte Carlo (QMC) rule for f and an N-element point set $\mathcal{P} = \{\boldsymbol{x}_0, \dots, \boldsymbol{x}_{N-1}\}$, i.e., $Q_N(f) = \frac{1}{N}\sum_{n=0}^{N-1} f(\boldsymbol{x}_n)$.
Prob	Probability.
$\mathbb{E}$	Expectation.
Var	Variance.
$\lvert\boldsymbol{x}\rvert_1$	L_1-norm; $\lvert\boldsymbol{x}\rvert_1 = \lvert x_1\rvert + \cdots + \lvert x_s\rvert$ if $\boldsymbol{x} = (x_1, \dots, x_s)$.
$\lvert\boldsymbol{x}\rvert_\infty$	Maximum norm; $\lvert\boldsymbol{x}\rvert_\infty = \max_{1\leq i\leq s} \lvert x_i\rvert$ if $\boldsymbol{x} = (x_1, \dots, x_s)$.

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Index